

VERLINDE-TYPE FORMULAS FOR RATIONAL SURFACES

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ABSTRACT. For a projective algebraic surface X , with an ample line bundle ω , let $M_\omega^X(c_1, d)$ be the moduli space of rank 2, ω -semistable torsion free sheaves E with $c_1(E) = c_1$ and $4c_2(E) - c_1^2 = d$. For line bundles L on X , let $\mu(L)$ be the corresponding determinant line bundles on $M_H^X(c_1, d)$. The K -theoretic Donaldson invariants are the holomorphic Euler characteristics $\chi(M_\omega^X(c_1, d), \mu(L))$. In this paper we develop an algorithm which in principle determines all their generating functions for the projective plane, its blowup in finitely many points, and also for $\mathbb{P}^1 \times \mathbb{P}^1$. Among others, we apply this algorithm to compute the generating functions of the $\chi(M_H^{\mathbb{P}^2}(0, d), \mu(nH))$ and $\chi(M_H^{\mathbb{P}^2}(H, d), \mu(nH))$ for $n \leq 11$, for H the hyperplane class on \mathbb{P}^2 . We give some conjectures about the general structure of these generating functions and interpret them in terms of Le Potier's strange duality conjecture.

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1. INTRODUCTION

In this paper let (X, ω) be a pair of a rational surface X and an ample line bundle ω . We consider the moduli spaces $M_\omega^X(c_1, d)$ of ω -semistable torsion-free coherent sheaves of rank 2 on X with Chern classes $c_1 \in H^2(X, \mathbb{Z})$ and c_2 such that $d = 4c_2 - c_1^2$. Associated to a line bundle L on X there is a determinant bundle $\mu(L) \in \text{Pic}(M_\omega^X(c_1, d))$. If L is ample, then $\mu(L)$ is nef and big on $M_\omega^X(c_1, d)$, and a suitable power induces the map from $M_\omega^X(c_1, d)$ to the corresponding Uhlenbeck compactification.

If one considers instead of a rational surface X a curve C , the spaces of sections of the corresponding determinant bundles are the spaces of conformal blocks, and their dimensions are given by the celebrated Verlinde formula. In [23] many reformulations of this formula are given. In particular [Thm. 1.(vi)][23] expresses the generating function on a fixed curve as a rational function. In this paper we study the generating functions of the holomorphic Euler characteristics $\chi(M_\omega^X(c_1, d), \mu(L))$, and show that they are given as rational functions. Let

$$\chi_{c_1}^{X, \omega}(L) := \sum_{d > 0} \chi(M_\omega^X(c_1, d), \mu(L)) \Lambda^d$$

(In case $c_1 = 0$ the coefficient of Λ^4 is slightly different, furthermore in case ω lies on a wall (see below), here instead of $\chi(M_\omega^X(c_1, d), \mu(L))$ we use the average over the chambers adjacent to ω .) We can view the spaces of sections $H^0(M_\omega^X(c_1, d), \mu(L))$ as analogues to the spaces of conformal blocks. In most cases we will consider (see Proposition 2.9 below), the higher cohomology groups of the determinant bundle $\mu(L)$ vanish. Thus our formulas for the $\chi_{c_1}^{X, \omega}(L)$ are analogues of the Verlinde formula for rational surfaces.

Notation 1.1. For two Laurent series $P(\Lambda) = \sum_n a_n \Lambda^n, Q(\Lambda) = \sum_n b_n \Lambda^n \in \mathbb{Q}[\Lambda^{-1}][[\Lambda]]$ we write $P(\Lambda) \equiv Q(\Lambda)$ if there is an $n_0 \in \mathbb{Z}$ with $a_n = b_n$ for all $n \geq n_0$.

Theorem 1.2. *Let X be \mathbb{P}^2 , $\mathbb{P}^1 \times \mathbb{P}^1$ or a blowup of \mathbb{P}^2 in n points. Let $c_1 \in H^2(X, \mathbb{Z})$, $L \in \text{Pic}(X)$. There is a polynomial $P_{c_1, L}^X(\Lambda) \in \Lambda^{-c_1^2} \mathbb{Q}[\Lambda^{\pm 4}]$ and $l_{c_1, L}^X \in \mathbb{Z}_{\geq 0}$, such that*

$$\chi_{c_1}^{X, \omega}(L) \equiv \frac{P_{c_1, L}^X(\Lambda)}{(1 - \Lambda^4)^{l_{c_1, L}^X}}.$$

Here ω is an ample line bundle on X with $\langle \omega, K_X \rangle < 0$. In case X is the blowup of \mathbb{P}^2 in n points we assume furthermore that $\omega = H - a_1 E_1 - \dots - a_n E_n$, with $|a_i| < \frac{1}{\sqrt{n}}$ for all i . Note that $P_{c_1, L}^X(\Lambda)$, $l_{c_1, L}^X$ are independent of ω (subject to the conditions above). In particular for any other ample line bundle ω' on X satisfying the conditions for ω above, we have $\chi_{c_1}^{X, \omega'}(L) - \chi_{c_1}^{X, \omega}(L) \in \mathbb{Z}[\Lambda]$.

We will see that there is an algorithm for determining the generating functions $\chi_{c_1}^{X, \omega}(L)$ of Theorem 1.2. Let now H be the hyperplane bundle on \mathbb{P}^2 . We apply the algorithm above to determine the generating functions of the $\chi(M_H^{\mathbb{P}^2}(0, d), \mu(nH))$ and the $\chi(M_H^{\mathbb{P}^2}(H, d), \mu(nH))$ for $n \leq 11$. These were determined before (and strange duality proven) for $c_1 = 0$ and $n = 1, 2$ in [1], and for all c_1 for $n = 1, 2, 3$ in [10]. We get the following result. Put

$$\begin{aligned} p_1(t) &= p_2(t) = 1, \quad p_3(t) = 1 + t^2, \quad p_4(t) = 1 + 6t^2 + t^3, \quad p_5(t) = 1 + 21t^2 + 20t^3 + 21t^4 + t^6, \\ p_6(t) &= 1 + 56t^2 + 147t^3 + 378t^4 + 266t^5 + 148t^6 + 27t^7 + t^8, \\ p_7(t) &= 1 + 126t^2 + 690t^3 + 3435t^4 + 7182t^5 + 9900t^6 + 7182t^7 + 3435t^8 + 690t^9 + 126t^{10} + t^{12}, \\ p_8(t) &= 1 + 252t^2 + 2475t^3 + 21165t^4 + 91608t^5 + 261768t^6 + 462384t^7 + 549120t^8 + 417065t^9 \\ &\quad + 210333t^{10} + 66168t^{11} + 13222t^{12} + 1515t^{13} + 75t^{14} + t^{15}, \\ p_9(t) &= 1 + 462t^2 + 7392t^3 + 100359t^4 + 764484t^5 + 3918420t^6 + 13349556t^7 + 31750136t^8 \\ &\quad + 52917800t^9 + 62818236t^{10} + 52917800t^{11} + 31750136t^{12} + 13349556t^{13} + 3918420t^{14} \\ &\quad + 764484t^{15} + 100359t^{16} + 7392t^{17} + 462t^{18} + t^{20}, \\ p_{10}(t) &= 1 + 792t^2 + 19305t^3 + 393018t^4 + 4788696t^5 + 39997980t^6 + 231274614t^7 + 961535355t^8 \\ &\quad + 2922381518t^9 + 6600312300t^{10} + 11171504661t^{11} + 14267039676t^{12} + 13775826120t^{13} \\ &\quad + 10059442536t^{14} + 5532629189t^{15} + 2277448635t^{16} + 693594726t^{17} + 154033780t^{18} + 24383106t^{19} \\ &\quad + 2669778t^{20} + 192588t^{21} + 8196t^{22} + 165t^{23} + t^{24}, \\ p_{11}(t) &= 1 + 1287t^2 + 45474t^3 + 1328901t^4 + 24287340t^5 + 309119723t^6 + 2795330694t^7 \\ &\quad + 18571137585t^8 + 92530378876t^9 + 351841388847t^{10} + 1033686093846t^{11} + 2369046974245t^{12} \\ &\quad + 4264149851544t^{13} + 6056384937603t^{14} + 6805690336900t^{15} + 6056384937603t^{16} + 4264149851544t^{17} \\ &\quad + 2369046974245t^{18} + 1033686093846t^{19} + 351841388847t^{20} + 92530378876t^{21} + 18571137585t^{22} \\ &\quad + 2795330694t^{23} + 309119723t^{24} + 24287340t^{25} + 1328901t^{26} + 45474t^{27} + 1287t^{28} + t^{30}. \end{aligned}$$

For $1 \leq n \leq 11$ we put $P_n(\Lambda) := p_n(\Lambda^4)$, $Q_n(\Lambda) = \Lambda^{n^2-1} P_n(\frac{1}{\Lambda})$. It is easy to see that for n odd, P_n is symmetric, i.e. $P_n(\Lambda) = Q_n(\Lambda)$.

Theorem 1.3. *For $1 \leq n \leq 11$ we have*

(1)

$$1 + \binom{n+2}{2} \Lambda^4 + \sum_{d>4} \chi(M_H^{\mathbb{P}^2}(0, d), \mu(nH)) \Lambda^d = \frac{P_n(\Lambda)}{(1 - \Lambda^4)^{\binom{n+2}{2}}}.$$

(2) if n is even, then

$$\sum_{d>0} \chi(M_H^{\mathbb{P}^2}(H, d), \mu(nH)) \Lambda^d = \frac{Q_n(\Lambda)}{(1 - \Lambda^4)^{\binom{n+2}{2}}}.$$

We see that for $n \leq 11$, the generating functions $\chi_H^{\mathbb{P}^2, H}(nh)$, $\chi_0^{\mathbb{P}^2, H}(nH)$ have a number of interesting features, which we conjecture to hold for all $n > 0$.

Conjecture 1.4. For all $n > 0$ there are polynomials $p_n(t) \in \mathbb{Z}[t]$ such the following holds. We put $P_n(\Lambda) = p_n(\Lambda^4)$, $Q_n(\Lambda) = \Lambda^{n^2-1} P_n(\frac{1}{\Lambda})$.

(1)

$$1 + \binom{n+2}{2} \Lambda^4 + \sum_{d>4} \chi(M_H^{\mathbb{P}^2}(0, d), \mu(nH)) \Lambda^d = \frac{P_n(\Lambda)}{(1 - \Lambda^4)^{\binom{n+2}{2}}}.$$

(2) If n is odd, then $P_n(\Lambda) = Q_n(\Lambda)$, if n is even then

$$\sum_{d>0} \chi(M_H^{\mathbb{P}^2}(H, d), \mu(nH)) \Lambda^d = \frac{Q_n(\Lambda)}{(1 - \Lambda^4)^{\binom{n+2}{2}}}.$$

(3) $p_n(1) = 2^{\binom{n-1}{2}}$.(4) For i odd and $i \leq n - 3$ we have

$$\text{Coeff}_{x^i} [e^{-(n^2-1)x/2} P_n(e^x)] = \text{Coeff}_{x^i} [e^{-(n^2-1)x/2} Q_n(e^x)] = 0.$$

(5) The degree of $p_n(t)$ is the largest integer strictly smaller than $n^2/4$.

On $\mathbb{P}^1 \times \mathbb{P}^1$ we get the following results. Let F and G be the classes of the fibres of the projections to the two factors. Let

$$\begin{aligned} q_1^0 &:= 1, \quad q_2^0 := 1 + t^2, \quad q_3^0 = 1 + 10t^2 + 4t^3 + t^4, \quad q_4^0 := 1 + 46t^2 + 104t^3 + 210t^4 + 104t^5 + 46t^6 + t^8, \\ q_5^0 &:= 1 + 146t^2 + 940t^3 + 5107t^4 + 12372t^5 + 19284t^6 + 16280t^7 + 8547t^8 + 2452t^9 + 386t^{10} + 20t^{11} + t^{12}, \\ q_6^0 &:= 1 + 371t^2 + 5152t^3 + 58556t^4 + 361376t^5 + 1469392t^6 + 3859616t^7 + 6878976t^8 + 8287552t^9 \\ &\quad + 6878976t^{10} + 3859616t^{11} + 1469392t^{12} + 361376t^{13} + 58556t^{14} + 5152t^{15} + 371t^{16} + t^{18}, \\ q_7^0 &:= 1 + 812t^2 + 20840t^3 + 431370t^4 + 5335368t^5 + 44794932t^6 + 259164216t^7 + 1070840447t^8 \\ &\quad + 3214402272t^9 + 7125238944t^{10} + 11769293328t^{11} + 14581659884t^{12} + 13577211024t^{13} \\ &\quad + 9496341984t^{14} + 4966846032t^{15} + 1928398719t^{16} + 548923040t^{17} + 112654644t^{18} + 16232904t^{19} \\ &\quad + 1584906t^{20} + 97448t^{21} + 3564t^{22} + 56t^{23} + t^{24}, \\ q_2^{F+G} &= t^{\frac{1}{2}}(1 + t), \quad q_4^{F+G} = t^{\frac{1}{2}}(1 + 10t + 84t^2 + 161t^3 + 161t^4 + 84t^5 + 10t^6 + t^7), \\ q_6^{F+G} &= t^{\frac{1}{2}}(1 + 35t + 1296t^2 + 18670t^3 + 154966t^4 + 770266t^5 + 2504382t^6 + 5405972t^7 + 7921628t^8 \\ &\quad + 7921628t^9 + 5405972t^{10} + 2504382t^{11} + 770266t^{12} + 154966t^{13} + 18670t^{14} + 1296t^{15} + 35t^{16} + t^{17}), \end{aligned}$$

$$\begin{aligned}
 q_2^F &:= 2t, \quad q_4^F := 3t + 43t^2 + 105t^3 + 210t^4 + 105t^5 + 43t^6 + 3t^7, \\
 q_6^F &:= 4t + 274t^2 + 5520t^3 + 57022t^4 + 366052t^5 + 1460922t^6 + 3873184t^7 + 6855798t^8 + 8316880t^9 \\
 &\quad + 6855798t^{10} + 3873184t^{11} + 1460922t^{12} + 366052t^{13} + 57022t^{14} + 5520t^{15} + 274t^{16} + 4t^{17}.
 \end{aligned}$$

For n odd put $q_n^{F+G}(t) = t^{d^2/2} q_n^0(t^{-1})$. Then we get

Theorem 1.5. (1) $\sum_{d>4} \chi(M_{F+G}^{\mathbb{P}^1 \times \mathbb{P}^1}(0, d), \mu(nF + nG)) \Lambda^d = \frac{q_n^0(\Lambda^4)}{(1 - \Lambda^4)^{(n+1)^2}} - 1 - (n^2 + 2n + 1) \Lambda^4$
for $1 \leq n \leq 7$.

(2) $\sum_{d>0} \chi(M_{F+G}^{\mathbb{P}^1 \times \mathbb{P}^1}(F + G, d), \mu(nF + nG)) \Lambda^d = \frac{q_n^{F+G}(\Lambda^4)}{(1 - \Lambda^4)^{(n+1)^2}} - \Lambda^2$ for $1 \leq n \leq 7$.

(3) $\sum_{d>0} \chi(M_{F+G}^{\mathbb{P}^1 \times \mathbb{P}^1}(F, d), \mu(nF + nG)) \Lambda^d = \frac{q_n^F(\Lambda^4)}{(1 - \Lambda^4)^{(n+1)^2}}$ for $n = 2, 4, 6$.

Remark 1.6. (1) For d even and $c_1 = 0$, F , $F + G$ we have $q_n^{c_1}(t) = t^{n^2/2} q_n^{c_1}(t^{-1})$.

(2) For all $1 \leq n \leq 7$ we have $q_n^0(1) = q_n^{F+G}(1) = 2^{(n-1)^2}$, and if d is also even $q_n^F(1) = 2^{(n-1)^2}$.

(3) For all $1 \leq n \leq 7$ and all i odd with $i \leq n - 2$ we have $\text{Coeff}_{x^i} [e^{-n^2 x/4} q_d^0(e^x)] = \text{Coeff}_{x^i} [e^{-n^2 x/4} q_n^{F+G}(e^x)] = 0$.

The results on \mathbb{P}^2 , $\mathbb{P}^1 \times \mathbb{P}^1$ as well as the computations for other rational surfaces lead to a general conjecture. For a line bundle L on a rational surface X we denote $\chi(L) = L(L - K_X)/2 + 1$ the holomorphic Euler characteristic and $g(L) = L(L + K_X)/2 + 1$ the genus of a smooth curve in the linear system $|L|$.

Conjecture 1.7. Let X be a rational surface and let ω be ample on X with $\langle \omega, K_X \rangle < 0$. Let L be a sufficiently ample line bundle on X . Then we have the following.

(1) There is a polynomial $P_{c_1, L}^X(\Lambda) \in \Lambda^{-c_1^2} \mathbb{Z}_{\geq 0}[\Lambda^{\pm 4}]$, such that

$$\sum_{d \geq 0} \chi(M_{\omega}^X(c_1, d), \mu(L)) \Lambda^d \equiv \frac{P_{c_1, L}^X(\Lambda)}{(1 - \Lambda^4)^{\chi(L)}}.$$

(2) We have $P_{c_1, L}^X(1) = 2^{g(L)}$.

(3) We have the "duality"

$$P_{c_1, L}^X(\Lambda) = \Lambda^{L^2 + 8 - K_X^2} P_{L + K_X - c_1, L}^X\left(\frac{1}{\Lambda}\right).$$

(4) If i is odd, and L is sufficiently ample with respect to i , then

$$\text{Coeff}_{x^i} [e^{-\frac{1}{2}(L^2 + 8 - K_X^2)x} P_{c_1, L}^X(e^x)] = 0.$$

In the case of (\mathbb{P}^2, dH) and $(\mathbb{P}^1 \times \mathbb{P}^1, dF + dG)$ sufficiently ample with respect to i means that $L + K_X$ is i -very ample.

Remark 1.8. The polynomial $P_{c_1,L}^X(\Lambda)$ is not well defined. We can write $P_{c_1,L}^X(\Lambda) = \Lambda^{-c_1^2} p_{c_1,L}^X(\Lambda^4)$, and the polynomial $p_{c_1,L}^X(t)$ is well defined only up to adding a Laurent polynomial in t divisible by $(1-t)^{\chi(L)}$. On the other hand, if L is sufficiently ample with respect to c_1, X , we conjecture that we can choose $p_{c_1,L}^X(t)$ with $\deg(p_{c_1,L}^X(t)) < \chi(L)$ (i.e. the difference in degree of the highest order and lowest order term in $p_{c_1,L}^X(t)$ is smaller than $\chi(L)$). Assuming this, $p_{c_1,L}^X(t)$ and thus $P_{c_1,L}^X(\Lambda)$ are uniquely determined.

Remark 1.9. Part (1) of Conjecture 1.7 requires a condition of sufficient ampleness (see Theorem 8.15). On the other hand it appears that a modified version of the conjecture holds in larger generality, i.e. $\chi_{X,c_1}^\omega(L) \equiv \frac{P_{c_1,L}^X(\Lambda)}{(1-\Lambda^4)^{\chi(L)}}$ with $P_{c_1,L}^X(\Lambda) \in \Lambda^{-c_1^2} \mathbb{Q}[\Lambda^{\pm 4}]$, and

- (1) $P_{c_1,L}^X(1) = 2^{g(L)}$,
- (2) $\chi_{X,c_1}^\omega(L) \equiv (-1)^{\chi(L)} \Lambda^{-(L-K_X)^2+4} \cdot \chi_{X,L+K_X-c_1}^\omega(L)|_{\Lambda=\frac{1}{\Lambda}}.$

Approach. This paper is built on [10], and both papers are built on [9]. In [9] the wallcrossing terms for the K -theoretic Donaldson invariants are determined in terms of modular forms, based on the solution of the Nekrasov conjecture for the K -theoretic partition function (see [20], [21], [17],[18],[19]). , and both [10] and this paper sum up the wallcrossing terms to get closed formulas for the generating functions. The main new inputs are the systematic use of the generating function the " K -theoretic Donaldson invariants with point class" $\chi_{c_1}^{X,\omega}(L, P^r)$, and the blowup formulas. We introduce in an ad hoc way $\chi_{c_1}^{X,\omega}(L, P^r) := \frac{1}{\Lambda^r} \hat{\chi}_{c_1+E}^{\hat{X},\omega}(L-E)$, where \hat{X} is the blowup of X in r general points and E is the sum of the exceptional divisors (but note that these are invariants on X , depending on an ample class ω on X). These invariants satisfy a wallcrossing formula which is very similar to that of the standard K -theoretic Donaldson invariants $\chi_{c_1}^{X,\omega}(L)$. We prove blowup formulas that compute all the generating formulas of K -theoretic Donaldson invariants on any blowup of X in terms of the $\chi_{c_1}^{X,\omega}(L, P^r)$. On the other hand we also prove blowdown formulas, which compute all the generating functions of the K -theoretic Donaldson invariants with point class $\chi_{c_1}^{X,\omega}(M, P^r)$ in terms of a *very small part* of those on the blowup \hat{X} . Then, generalizing the methods of [10], we compute this small part in the case \hat{X} is the blowup of \mathbb{P}^2 in a point. Thus, using the blowdown formulas, we determine the generating functions of the K theoretic Donaldson invariants with point class of \mathbb{P}^2 , and thus, by using the blowup formula again, of all blowups of \mathbb{P}^2 . Finally, as the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ in a point is equal to the blowup of \mathbb{P}^2 in two points, we apply the blowdown formulas again to determine generating functions for $\mathbb{P}^1 \times \mathbb{P}^1$. These methods give an algorithm, which in principle computes all the generating functions mentioned above. The algorithm proves the rationality of the generating functions, and is carried for many X and L to obtain the explicit generating functions $\chi_{c_1}^{X,\omega}(L)$.

2. BACKGROUND MATERIAL

In this whole paper X will be a simply connected nonsingular projective rational surface over \mathbb{C} . Usually X will be \mathbb{P}^2 , $\mathbb{P}^1 \times \mathbb{P}^1$ or the blowup of \mathbb{P}^2 in finitely many points.

We will fix some notation that we want to use during this whole paper.

- Notation 2.1.** (1) For a class $\alpha, \beta \in H^2(X, \mathbb{Q})$, we denote $\langle \alpha, \beta \rangle$ their intersection product. For $\beta \in H^2(X)$ we also write β^2 instead of $\langle \beta, \beta \rangle$.
- (2) For a line bundle L on X we denote its first Chern class by the same letter.
- (3) If \widehat{X} is the blowup of X in a point or in a finite set of points, and $L \in \text{Pic}(X)$, we denote its pullback to \widehat{X} by the same letter. The same holds for classes $\alpha \in H^2(X, \mathbb{R})$.
- (4) We denote $\mathcal{R} := \mathbb{Q}[[q^2\Lambda^2, q^4]]$.
- (5) Let $\mathbb{Q}[t_1, \dots, t_k]_n$ be the set of polynomials in t_1, \dots, t_k of degree n and $\mathbb{Q}[t_1, \dots, t_k]_{\leq n}$ the polynomials in t_1, \dots, t_n of degree at most n .
- (6) Let ω be an ample divisor on X . For $r \geq 0$, $c_1 \in \text{Pic}(X)$, $c_2 \in H^4(X, \mathbb{Z})$ let $M_\omega^X(r, c_1, c_2)$ the moduli space of ω -semistable rank r sheaves on X with $c_1(E) = c_1$, $c_2(E) = c_2$. Let $M_\omega^X(r, c_1, c_2)_s$ be the open subset of stable sheaves. We will write $M_\omega^X(c_1, d)$ with $d := 4c_2 - c_1^2$ instead of $M_\omega^X(2, c_1, c_2)$.

2.1. Determinant line bundles. We briefly review the determinant line bundles on the moduli space [8],[13], [14], for more details we refer to [12, Chap. 8]. We mostly follow [9, Sec. 1.1,1.2].

For a Noetherian scheme Y we denote by $K(Y)$ and $K^0(Y)$ the Grothendieck groups of coherent sheaves and locally free sheaves on Y respectively. If Y is nonsingular and quasiprojective, then $K(Y) = K^0(Y)$. If we want to distinguish a sheaf \mathcal{F} and its class in $K(Y)$, we denote the latter by $[\mathcal{F}]$. The product $[\mathcal{F}].[\mathcal{G}] := \sum_i (-1)^i [\text{Tor}_i(\mathcal{F}, \mathcal{G})]$ makes $K^0(Y)$ into a commutative ring and $K(Y)$ into a $K^0(Y)$ module. For a proper morphism $f: Y_1 \rightarrow Y_2$ we have the pushforward homomorphism $f_!: K(Y_1) \rightarrow K(Y_2)$; $[\mathcal{F}] \mapsto \sum_i (-1)^i [R^i f_* \mathcal{F}]$. For any morphism $f: Y_1 \rightarrow Y_2$ we have the pullback homomorphism $f^*: K^0(Y_2) \rightarrow K^0(Y_1)$ given by $[\mathcal{F}] \mapsto [f^* \mathcal{F}]$ for a locally free sheaf \mathcal{F} on Y_2 . Let \mathcal{E} be a flat family of coherent sheaves of class c on X parametrized by a scheme S , then $\mathcal{E} \in K^0(X \times S)$. Let $p: X \times S \rightarrow S$, $q: X \times S \rightarrow X$ be the projections. Define $\lambda_{\mathcal{E}}: K(X) \rightarrow \text{Pic}(S)$ as the composition of the following homomorphisms:

$$(2.2) \quad K(X) = K^0(X) \xrightarrow{q^*} K^0(X \times S) \xrightarrow{[\mathcal{E}]} K^0(X \times S) \xrightarrow{p_!} K^0(S) \xrightarrow{\det^{-1}} \text{Pic}(S),$$

By Proposition 2.1.10 in [12] $p_!([\mathcal{F}]) \in K^0(S)$ for \mathcal{F} S -flat. We have the following facts.

- (1) $\lambda_{\mathcal{E}}$ is a homomorphism, i.e. $\lambda_{\mathcal{E}}(v_1 + v_2) = \lambda_{\mathcal{E}}(v_1) \otimes \lambda_{\mathcal{E}}(v_2)$.
- (2) If $\mu \in \text{Pic}(S)$ is a line bundle, then $\lambda_{\mathcal{E} \otimes p^* \mu}(v) = \lambda_{\mathcal{E}}(v) \otimes \mu^{\chi(c \otimes v)}$.
- (3) $\lambda_{\mathcal{E}}$ is compatible with base change: if $\phi: S' \rightarrow S$ is a morphism, then $\lambda_{\phi^* \mathcal{E}}(v) = \phi^* \lambda_{\mathcal{E}}(v)$.

Define $K_c := c^\perp = \{v \in K(X) \mid \chi(v \otimes c) = 0\}$, and $K_{c,\omega} := c^\perp \cap \{1, h, h^2\}^{\perp\perp}$, where $h = [\mathcal{O}_\omega]$. Then we have a well-defined morphism $\lambda: K_c \rightarrow \text{Pic}(M_\omega^X(c)^s)$, and $\lambda: K_{c,\omega} \rightarrow \text{Pic}(M_\omega^X(c))$ satisfying the following properties:

- (1) The λ commute with the inclusions $K_{c,\omega} \subset K_c$ and $\text{Pic}(M_\omega^X(c)) \subset \text{Pic}(M_\omega^X(c)^s)$.

- (2) If \mathcal{E} is a flat family of semistable sheaves on X of class c parametrized by S , then we have $\phi_{\mathcal{E}}^*(\lambda(v)) = \lambda_{\mathcal{E}}(v)$ for all $v \in K_{c,\omega}$ with $\phi_{\mathcal{E}} : S \rightarrow M_{\omega}^X(c)$ the classifying morphism.
- (3) If \mathcal{E} is a flat family of stable sheaves, the statement of (2) holds with $K_{c,\omega}$, $M_{\omega}^X(c)$ replaced by K_c , $M_{\omega}^X(c)^s$.

Since X is a simply connected surface, both the moduli space $M_{\omega}^X(c)$ and the determinant line bundle $\lambda(c^*)$ only depend on the images of c and c^* in $K(X)_{num}$. Here $K(X)_{num}$ is the Grothendieck group modulo numerical equivalence. We say that $u, v \in K(X)$ are numerically equivalent if $u - v$ is in the radical of the quadratic form $(u, v) \mapsto \chi(X, u \otimes v) \equiv \chi(u \otimes v)$

We call H *general* with respect to c if all the strictly semistable sheaves in $M_H^X(c)$ are strictly semistable with respect to all ample divisors on X in a neighbourhood of H

Often $\lambda : K_{c,\omega} \rightarrow \text{Pic}(M_{\omega}^X(c))$ can be extended. For instance let $c = (2, c_1, c_2)$, then $\lambda(v(L))$ is well-defined over $M_{\omega}^X(c)$ if $\langle L, \xi \rangle = 0$ for all ξ a class of type (c_1, d) (see §2.2) with $\langle \omega, \xi \rangle = 0$. This can be seen from the construction of $\lambda(v(L))$ (e.g. see the proof of Theorem 8.1.5 in [12]).

2.2. Walls. Denote by $\mathcal{C} \subset H^2(X, \mathbb{R})$ the ample cone of X . Then \mathcal{C} has a chamber structure: For a class $\xi \in H^2(X, \mathbb{Z}) \setminus \{0\}$ let $W^{\xi} := \{x \in \mathcal{C} \mid \langle x, \xi \rangle = 0\}$. Assume $W^{\xi} \neq \emptyset$. Let $c_1 \in \text{Pic}(X)$, $d \in \mathbb{Z}$ congruent to $-c_1^2$ modulo 4. Then we call ξ a *class of type* (c_1, d) and call W^{ξ} a *wall of type* (c_1, d) if the following conditions hold

- (1) $\xi + c_1$ is divisible by 2 in $H^2(X, \mathbb{Z})$,
- (2) $d + \xi^2 \geq 0$.

We call ξ a *class of type* (c_1) , if $\xi + c_1$ is divisible by 2 in $H^2(X, \mathbb{Z})$. We say that $\omega \in \mathcal{C}$ lies on the wall W^{ξ} if $\omega \in W^{\xi}$. The *chambers of type* (c_1, d) are the connected components of the complement of the walls of type (c_1, d) in \mathcal{C} . Then $M_{\omega}^X(c_1, d)$ depends only on the chamber of type (c_1, d) of ω . Let $c \in K(X)$ be the class of $\mathcal{F} \in M_{\omega}^X(c_1, d)$. It is easy to see that ω is general with respect to c if and only if ω does not lie on a wall of type (c_1, d) .

2.3. K -theoretic Donaldson invariants. We write $M_{\omega}^X(c_1, d)$ for $M_{\omega}^X(2, c_1, c_2)$ with $d = 4c_2 - c_1^2$. Let $v \in K_c$, where c is the class of a coherent rank 2 sheaf with Chern classes c_1, c_2 . Let L be a line bundle on X and assume that $\langle L, c_1 \rangle$ is even. Then for c the class of a rank 2 coherent sheaf with Chern classes c_1, c_2 , we put

$$(2.3) \quad v(L) := (1 - L^{-1}) + \left\langle \frac{L}{2}, L + K_X + c_1 \right\rangle [\mathcal{O}_x] \in K_c.$$

Note that $v(L)$ is independent of c_2 . Assume that ω is general with respect to $(2, c_1, c_2)$. Then we denote $\mu(L) := \lambda(v(L)) \in \text{Pic}(M_{\omega}^X(c_1, d))$. The *K -theoretic Donaldson invariant* of X , with respect to L, c_1, d, ω is $\chi(M_{\omega}^X(c_1, d), \mathcal{O}(\mu(L)))$.

We recall the following blowup relation for the K -theoretic Donaldson invariants from [9, Sec.1.4]. Let (X, ω) be a polarized rational surface. Let \widehat{X} be the blowup of X in a point and E the exceptional divisor. In the following we always denote a class in $H^*(X, \mathbb{Z})$ and its pullback by the same letter. Let Q be an open subset of a suitable quot-scheme such that $M_{\omega}^X(c_1, d) = Q/GL(N)$. Assume that Q is smooth (e.g. $\langle -K_X, \omega \rangle > 0$). We choose $\epsilon > 0$

sufficiently small so that $\omega - \epsilon E$ is ample on \widehat{X} and there is no class ξ of type (c_1, d) or of type $(c_1 + E, d + 1)$ on \widehat{X} with $\langle \xi, \omega \rangle < 0 < \langle \xi, (\omega - \epsilon E) \rangle$. In case $c_1 = 0$ assume $d > 4$.

Lemma 2.4. *We have*

$$\begin{aligned}\chi(M_{\omega - \epsilon E}^{\widehat{X}}(c_1, d), \mu(L)) &= \chi(M_{\omega}^X(c_1, d), \mu(L)), \\ \chi(M_{\omega - \epsilon E}^{\widehat{X}}(c_1 + E, d + 1), \mu(L)) &= \chi(M_{\omega}^X(c_1, d), \mu(L))\end{aligned}$$

for any line bundle L on X such that $\langle L, c_1 \rangle$ is even and $\langle L, \xi \rangle = 0$ for ξ any class of type (c_1, d) on \widehat{X} with $\langle \omega, \xi \rangle = 0$.

Following [10], we introduce the generating function of the K -theoretic Donaldson invariants.

Definition 2.5. Let $c_1 \in H^2(X, \mathbb{Z})$. Let ω be ample on X not on a wall of type (c_1) .

(1) If $c_1 \notin 2H^2(X, \mathbb{Z})$, let

$$(2.6) \quad \chi_{c_1}^{X, \omega}(L) := \sum_{d > 0} \chi(M_{\omega}^X(c_1, d), \mathcal{O}(\mu(L))) \Lambda^d,$$

(2) In case $c_1 = 0$ let \widehat{X} be the blowup of X in a point. Let E be the exceptional divisor. Let $\epsilon > 0$ be sufficiently small so that there is no class ξ of class $(E, d + 1)$ on \widehat{X} with $\langle \xi, \omega \rangle < 0 < \langle \xi, (\omega - \epsilon E) \rangle$. We put

$$(2.7) \quad \chi_0^{X, \omega}(L) := \sum_{d > 4} \chi(M_{\omega}^X(0, d), \mathcal{O}(\mu(L))) \Lambda^d + \left(\chi(M_{\omega - \epsilon E}^{\widehat{X}}(E, 5), \mu(L)) + LK_X - \frac{K_X^2 + L^2}{2} - 1 \right) \Lambda^4.$$

Remark 2.8. ([9, Rem. 1.9]) If H is a general polarization, then $\mu(2K_X)$ is a line bundle on $M_H^X(c)$ which coincides with the dualizing sheaf on the locus of stable sheaves $M_H^X(c)^s$. If $\dim(M_H^X(c) \setminus M_H^X(c)^s) \leq \dim M_H^X(c) - 2$, then $\omega_{M_H^X(c)} = \mu(2K_X)$.

Under rather general assumptions the higher cohomology of $\mu(L)$ vanishes. The following follows from [10, Prop.2.9] and its proof, which is based on [9, Sec.1.4].

Proposition 2.9. *Fix c_1, d . Let ω be an ample line bundle on X which is general with respect to c_1, d , and satisfies $\langle -K_X, \omega \rangle > 0$. Let L be a nef line bundle on X such that $L - 2K_X$ is ample. If c_1 is not divisible by 2 in $H^2(X, \mathbb{Z})$ or $d > 8$, we have $H^i(M_{\omega}^X(c_1, d), \mu(L)) = 0$ for all $i > 0$, in particular*

$$\dim H^0(M_{\omega}^X(c_1, d), \mu(L)) = \chi(M_{\omega}^X(c_1, d), \mu(L)).$$

3. STRANGE DUALITY

3.1. Review of strange duality. We briefly review the strange duality conjecture from for surfaces from [15]. The strange duality conjecture was formulated for X a smooth curve in the 1990s (see [4] and [7]) and in this case been proved around 2007 (see [5], [16]). For X a surface, there is a formulation for some special due to Le Potier (see [15] or [6]). Let

$c, c^* \in K(X)_{num}$ with $c \in K_{c^*}$. Let H be an ample line bundle on X which is both c -general and c^* -general. Write $\mathcal{D}_{c,c^*} := \lambda(c^*) \in \text{Pic}(M_H^X(c))$, $\mathcal{D}_{c^*,c} := \lambda(c) \in \text{Pic}(M_H^X(c^*))$. Assume that all H -semistable sheaves \mathcal{F} on X of class c and all H -semistable sheaves \mathcal{G} on X of class c^* satisfy

- (1) $\underline{\text{Tor}}_i(\mathcal{F}, \mathcal{G}) = 0$ for all $i \geq 1$,
- (2) $H^2(X, \mathcal{F} \otimes \mathcal{G}) = 0$.

Both conditions are automatically satisfied if c is not of dimension 0 and c^* is of dimension 1 (see [15, p.9]).

Put $\mathcal{D} := \mathcal{D}_{c,c^*} \boxtimes \mathcal{D}_{c^*,c} \in \text{Pic}(M_H^X(c) \times M_H^X(c^*))$. In [15, Prop. 9] a canonical section σ_{c,c^*} of \mathcal{D} is constructed, whose zero set is supported on

$$\mathcal{D} := \{([\mathcal{F}], [\mathcal{G}]) \in M_H^X(c) \times M_H^X(c^*) \mid H^0(X, \mathcal{F} \otimes \mathcal{G}) \neq 0\}.$$

The element σ_{c,c^*} of $H^0(M_H^X(c), \mathcal{D}_{c,c^*}) \otimes H^0(M_H^X(c^*), \mathcal{D}_{c^*,c})$, gives a linear map

$$(3.1) \quad SD_{c,c^*} : H^0(M_H^X(c), \mathcal{D}_{c,c^*})^\vee \rightarrow H^0(M_H^X(c^*), \mathcal{D}_{c^*,c}),$$

called the *strange duality map*. Le Potier's strange duality conjecture is then the following.

Conjecture 3.2. Under the above assumptions SD_{c,c^*} is an isomorphism.

It seems natural to believe that under more general assumptions than Conjecture 3.2 we have the *numerical version of strange duality* $\chi(M_H^X(c), \mathcal{D}_{c,c^*})^\vee = \chi(M_H^X(c^*), \mathcal{D}_{c^*,c})$.

3.2. Interpretation of the main results and conjectures in view of strange duality.

In this subsection let $c = (2, c_1, c_2)$ and $c^* = (0, L, \chi = \langle \frac{L}{2}, c_1 \rangle)$, so that $\mathcal{D}_{c,c^*} = \mu(L)$. The moduli space $M_H^X(c^*)$ is a moduli space of pure dimension 1 sheaves. It has a natural projection $\pi := \pi^{L, c_1} : M_H^X(c^*) \rightarrow |L|$, whose fibre over a smooth curve C in $|L|$ is the Jacobian of line bundles degree $\langle \frac{L}{2}, c_1 + K_X + L \rangle$ on C . In particular c^* is independent of c_2 . In case $c_1 = 0$ the fibre of $\pi^{L, 0}$ over the class of a nonsingular curve C is the Jacobian $J_{g(C)-1}(C)$ of degree $g(C)-1 = \frac{1}{2} \deg(K_C)$ line bundles on C . In this case we denote by $\Theta := \lambda([\mathcal{O}_X]) \in \text{Pic}(M_H^X(c^*))$. The divisor of its restriction to a fibre $J_{g(C)-1}(C)$ is the classical theta divisor of degree $g(C)-1$ divisors on C with a section.

Let again c_1 be general and let $\mathcal{O}_X(c_1)$ be the line bundle with first Chern class c_1 ; we denote $\Theta_{2,c_1} := \lambda([\mathcal{O}_X \oplus \mathcal{O}_X(c_1)]) \in \text{Pic}(M_H^X(c^*))$. We also denote $\eta := \lambda(\mathcal{O}_x) \in \text{Pic}(M_H^X(c^*))$, for x a general point of X . It is standard that $\eta = \pi^*(\mathcal{O}_{|L|}(1))$, with $\mathcal{O}_{|L|}(1)$ the hyperplane bundle on $|L|$. Thus we see that $\mathcal{D}_{c^*,c} = \lambda(c) = \Theta_{2,c_1} \otimes \pi^*(\mathcal{O}_{|L|}(c_2))$; in particular in case $c_1 = 0$ we have $\mathcal{D}_{c^*,c} = \lambda(c) = \Theta^{\otimes 2} \otimes \pi^*(\mathcal{O}_{|L|}(c_2))$.

We use Le Potier's strange duality conjecture and the results and conjectures from the introduction to make conjectures about the pushforwards $\pi_*^{L, c_1}(\Theta_{2,c_1})$, $\pi_!^{L, c_1}(\Theta_{2,c_1})$. For a Laurent polynomial $f(t) := \sum_n a_n t^n \in \mathbb{Z}[t^{-1}, t]$ we put $f(\mathcal{O}_{|L|}(-1)) := \bigoplus_n \mathcal{O}_{|L|}(-n)^{\oplus a_n}$.

Conjecture 3.3. (1) If L is sufficiently ample on X , then, defining p_L^{X, c_1} as in Remark 1.8, then $\pi_*^{L, c_1}(\lambda(\Theta_{2,c_1})) \otimes \mathcal{O}_{|L|} = p_L^{X, c_1}(\mathcal{O}_{|L|}(-1))$ and $R^i \pi_*^{L, c_1}(\lambda(\Theta_{2,c_1})) = 0$ for $i > 0$. In

particular $\pi_*(\lambda_{c^*}(\Theta_{2,c_1}))$ splits as a direct sum of line bundles on $|L|$. (Note that this implies that p_L^{X,c_1} is a polynomial with nonnegative coefficients, as conjectured in Conjecture 1.7(1)).

- (2) In particular in the case $X = \mathbb{P}^2$, and $d > 0$, we get, with the polynomials $p_d(t)$ from Conjecture 1.4, that

$$\pi_*^{dH,0}(\lambda(\Theta^2)) = p_d(\mathcal{O}_{|dH|}(-1)), \quad \pi_*^{2dH,H}(\lambda(\Theta_{2,H})) = p_{2d}(\mathcal{O}_{|2dH|}(1)) \otimes \mathcal{O}_{|2dH|}(-d^2).$$

- (3) Under more general assumptions on L on X , we expect that there is a choice of $P_L^{X,c_1}(\Lambda) = \Lambda^{-c_1^2} p_L^{X,c_1}(\Lambda^4)$, such that $\pi_!^{L,c_1}(\lambda(\Theta_{2,c_1})) = p_L^{X,c_1}(\mathcal{O}_{|L|}(-1))$

Remark 3.4. (1) Assuming part (2) of Conjecture 3.3, Theorem 1.3 determines $\pi_*^{dH,0}(\lambda(\Theta^2))$, $\pi_*^{dH,H}(\lambda(\Theta_{2,H}))$ as direct sum of line bundles for $d \leq 11$.

- (2) For $X = \mathbb{P}^1 \times \mathbb{P}^1$, assuming part (1) of Conjecture 3.3, Theorem 1.5 gives, with the notation from there, for $d \leq 7$ that

$$\begin{aligned} \pi_*^{d(F+G),0}(\lambda(\Theta^2)) &= q_d^0(\mathcal{O}_{|d(F+G)|}(-1)), \\ \pi_*^{d(F+G),F}(\lambda(\Theta_{2,F})) &= q_d^F(\mathcal{O}_{|d(F+G)|}(-1)), \\ \pi_*^{d(F+G),F+G}(\lambda(\Theta_{2,F+G})) &= (t^{1/2} q_d^{F+G}(t))|_{t=(\mathcal{O}_{|d(F+G)|}(-1))}. \end{aligned}$$

- (3) In [10] some further generating functions for the K -theoretic Donaldson invariants of $X = \mathbb{P}^1 \times \mathbb{P}^1$ or $X = \tilde{\mathbb{P}}^2$ are computed. From the results there we expect

$$\begin{aligned} \pi_*^{nF+2G,0}(\Theta^2) &= (\mathcal{O}_{|nF+2G|} \oplus \mathcal{O}_{|nF+2G|}(-1))^{\otimes n}_{ev}, \\ \pi_*^{nF+2G,F}(\Theta_{2,F}) &= (\mathcal{O}_{|nF+2G|} \oplus \mathcal{O}_{|nF+2G|}(-1))^{\otimes n}_{odd}, \end{aligned}$$

where $(\bullet)_{ev}$ and $(\bullet)_{odd}$ denotes respectively the part consisting only of even powers $\mathcal{O}(-2d)$ or odd powers $\mathcal{O}(-2d-1)$. In particular this would give

$$\pi_*^{nF+2G,0}(\Theta^2) \oplus \pi_*^{nF+2G,F}(\Theta_{2,F}) = (\mathcal{O}_{|nF+2G|} \oplus \mathcal{O}_{|nF+2G|}(-1))^{\otimes n}.$$

Remark 3.5. We briefly motivate the above conjectures. Assuming strange duality Conjecture 3.2, we have, using also the projection formula,

$$\begin{aligned} H^0(M_\omega^X(c_1, d), \chi(L))^\vee &= H^0(M_\omega^X(c^*), \lambda(c)) = H^0(|L|, \pi_*^{L,c_1}(\lambda(c))) \\ &= H^0(|L|, \pi_*^{L,c_1}(\lambda(\Theta_{2,c_1})) \otimes \mathcal{O}_{|L|}(c_2)), \end{aligned}$$

and similarly, assuming the numerical version of strange duality above,

$$\chi(M_\omega^X(c_1, d), \chi(L)) = \chi(\lambda(c)) = \chi(M_\omega^X(c^*), \pi_!(\lambda(c))) = \chi(\pi_!^{L,c_1}(\lambda(\Theta_{2,c_1})) \otimes \mathcal{O}_{|L|}(c_2)).$$

We assume $H^i(X, L) = 0$ for $i > 0$, thus $\dim(|L|) = \chi(L) - 1$, then for $0 \leq l \leq \dim(|L|)$, and $n \geq 0$, we have

$$\sum_{n \geq 0} \chi(|L|, \mathcal{O}_{|L|}(-l+n)) t^n = \frac{t^l}{(1-t)^{\chi(L)}}.$$

Thus, assuming the numerical part of the strange duality conjecture and part (3) of Conjecture 3.3, we would get

$$\begin{aligned}\chi_{c_1}^{X,\omega}(L) &\equiv \Lambda^{-c_1^2} \sum_{n \geq 0} \chi(|L|, \pi_!^{L,c_1}(\Theta_{2,c_1}) \otimes \mathcal{O}_{|L|}(n)) \Lambda^{4n} \\ &\equiv \Lambda^{-c_1^2} \sum_{n \geq 0} \chi(|L|, p_L^{X,c_1}(\mathcal{O}_{|L|}(-1)) \otimes \mathcal{O}_{|L|}(n)) \Lambda^{4n} \\ &= \Lambda^{-c_1^2} \frac{p_L^{X,c_1}(\Lambda^4)}{(1 - \Lambda^4)^{\chi(L)}} = \frac{P_L^{X,c_1}(\Lambda)}{(1 - \Lambda^4)^{\chi(L)}}\end{aligned}$$

Assuming the strange duality conjecture and part (1) of Conjecture 3.3, we would get the same statement with the left hand side replaced by $\sum_{n \geq 0} H^0(M_H^X(2, c_1, n), \mu(L)) t^n$. In other words Conjecture 3.3 explains the generating functions of Theorem 1.3, Theorem 1.5 and Conjecture 1.7(1).

Remark 3.6. Assuming Conjecture 3.3 and the strange duality conjecture, we see that $\text{rk}(\pi_!(\Theta_{2,c_1})) = p_L^{X,c_1}(1)$. As mentioned above, the fibre over $\pi^{L,c_1} : M_H^X(c^*) \rightarrow |L|$ over the point corresponding to a smooth curve C in $|L|$ is the Jacobian $J_d(C)$ of line bundles degree $d = \langle \frac{L}{2}, c_1 + K_X + L \rangle$ on C , and we see that Θ_{2,c_1} is a polarisation of type $(2, \dots, 2)$. Thus by the Riemann-Roch theorem we have $\chi(J_d(C), \Theta_{2,c_1}|_{J_d(C)}) = 2^{g(C)}$. Thus Conjecture 3.3 implies that $\pi_!(\Theta_{2,c_1})$ has rank $2^{g(C)}$, therefore, assuming the strange duality conjecture, it implies $p_L^{X,c_1}(1) = 2^{g(C)}$, as predicted in Conjecture 1.7 and seen e.g. in Theorem 1.3 and Theorem 1.5 for many L in the case of \mathbb{P}^2 and $\mathbb{P}^1 \times \mathbb{P}^1$.

Remark 3.7. Let L again be sufficiently ample on X . Assuming the strange duality conjecture Conjecture 3.2 and part (1) of Conjecture 3.3 we get that part (3) of Conjecture 1.7 gives the conjectural duality

$$\pi_*^{L,c_1}(\Theta_{2,c_1}) = (\pi_*^{L,L+K_X-c_1}(\Theta_{2,L+K_X-c_1}))^\vee \otimes \mathcal{O}_{|L|}(-\langle L, L+K_X \rangle/2 - \langle c_1, c_1-K_X \rangle/2 + \langle L, c_1 \rangle/2 - 2).$$

In particular in case $c_1 = 0$,

$$\pi_*^{L,0}(\Theta^{\otimes 2}) = (\pi_*^{L,L+K_X}(\Theta_{2,L+K_X}))^\vee \otimes \mathcal{O}_{|L|}(-\langle L, L+K_X \rangle/2 - 2).$$

In the case of $X = \mathbb{P}^2$ we should have for $d > 0$ that

$$\begin{aligned}\pi_*^{2dH,0}(\Theta^{\otimes 2}) &= (\pi_*^{2dH,H}(\Theta_{2,H}))^\vee \otimes \mathcal{O}_{|2dH|}(-d^2), \\ \pi_*^{(2d+1)H,0}(\Theta^{\otimes 2}) &= (\pi_*^{(2d+1)H,0}(\Theta^{\otimes 2}))^\vee \otimes \mathcal{O}_{|(2d+1)H|}(-d(d+1)).\end{aligned}$$

Similarly we conjecture for $X = \mathbb{P}^1 \times \mathbb{P}^1$ e.g. that for $d > 0$

$$\pi_*^{2d(F+G),0}(\Theta^{\otimes 2}) = (\pi_*^{2d(F+G),0}(\Theta^{\otimes 2}))^\vee \otimes \mathcal{O}_{|2d(F+G)|}(-2d^2).$$

4. WALLCROSSING FORMULA

4.1. Theta functions and modular forms. We start by reviewing results and notations from [9], [10, Sec. 3.1]. For $\tau \in \mathcal{H} = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$ put $q = e^{\pi i \tau / 4}$ and for $h \in \mathbb{C}$ put $y = e^{h/2}$. Note that the notation is not standard. Recall the 4 Jacobi theta functions:

$$(4.1) \quad \begin{aligned} \theta_1(h) &:= \sum_{n \in \mathbb{Z}} i^{2n-1} q^{(2n+1)^2} y^{2n+1} = -iq(y - y^{-1}) \prod_{n>0} (1 - q^{8n})(1 - q^{8n}y^2)(1 - q^{8n}y^{-2}), \\ \theta_2(h) &:= \sum_{n \in \mathbb{Z}} q^{(2n+1)^2} y^{2n+1} = -q(y + y^{-1}) \prod_{n>0} (1 - q^{8n})(1 + q^{8n}y^2)(1 + q^{8n}y^{-2}), \\ \theta_3(h) &:= \sum_{n \in \mathbb{Z}} q^{(2n)^2} y^{2n}, \quad \theta_4(h) := \sum_{n \in \mathbb{Z}} i^{2n} q^{(2n)^2} y^{2n}. \end{aligned}$$

We usually do not write the argument τ . The conventions are essentially the same as in [22] and in [2], where the θ_i for $i \leq 3$ are denoted ϑ_i and θ_4 is denoted ϑ_0 . Denote

$$(4.2) \quad \begin{aligned} \theta_i &:= \theta_i(0), \quad \tilde{\theta}_i(h) := \frac{\theta_i(h)}{\theta_i}, \quad i = 2, 3, 4; \quad \tilde{\theta}_1(h) := \frac{\theta_1(h)}{\theta_4}, \\ u &:= -\frac{\theta_2^2}{\theta_3^2} - \frac{\theta_3^2}{\theta_2^2} = -\frac{1}{4}q^{-2} - 5q^2 + \frac{31}{2}q^6 - 54q^{10} + O(q^{14}), \end{aligned}$$

and two Jacobi functions, i.e. Jacobi forms of weight and index 0, $\Lambda := \frac{\theta_1(h)}{\theta_4(h)}$, $M := 2\frac{\tilde{\theta}_2(h)\tilde{\theta}_3(h)}{\theta_4(h)^2}$, which satisfy the relation

$$(4.3) \quad M = 2\sqrt{1 + u\Lambda^2 + \Lambda^4},$$

and the formulas

$$(4.4) \quad \frac{\partial \Lambda}{\partial h} = \frac{\theta_2 \theta_3}{4i} M, \quad h = \frac{2i}{\theta_2 \theta_3} \int_0^\Lambda \frac{dx}{\sqrt{1 + ux^2 + x^4}}.$$

In [10, Sec. 3.1] it is shown that $h \in iq^{-1}\Lambda\mathcal{R}$. A function $F(\tau, h)$ can via formula (4.4) also be viewed as a function of τ and Λ . In this case, viewing τ and Λ as the independent variables we define $F' := \frac{4}{\pi i} \frac{\partial F}{\partial \tau} = q \frac{\partial F}{\partial q}$, $F^* := \Lambda \frac{\partial F}{\partial \Lambda}$, and get

$$(4.5) \quad h^* = \frac{4i\Lambda}{\theta_2 \theta_3 M}, \quad u' = \frac{2\theta_4^8}{\theta_2^2 \theta_3^2}.$$

For future use we record the following standard formulas for the behaviour of the theta functions under translation.

$$(4.6) \quad \theta_4(h + 2\pi i) = \theta_4(h), \quad \theta_4(h + 2\pi i\tau) = -q^{-4}y^{-2}\theta_4(h), \quad \theta_4(h + \pi i\tau) = iq^{-1}y^{-1}\theta_1(h)$$

$$(4.7) \quad \theta_1(h + 2\pi i) = -\theta_1(h), \quad \theta_1(h + 2\pi i\tau) = -q^{-4}y^{-2}\theta_1(h), \quad \theta_1(h + \pi i\tau) = iq^{-1}y^{-1}\theta_4(h),$$

$$(4.8) \quad \theta_2(h + \pi i\tau) = q^{-1}y^{-1}\theta_3(h), \quad \theta_3(h + \pi i\tau) = q^{-1}y^{-1}\theta_2(h).$$

(see e.g. [2, Table VIII, p. 202]).

Lemma 4.9. *Let $a, b \in \mathbb{Z}$. Then*

$$(1) \quad \theta_4(h) = (-1)^b q^{4b^2} y^{2b} \theta_4(h + 2\pi i b \tau), \quad \theta_4(h + 2\pi i a) = \theta_4(z),$$

- (2) $\theta_1(h) = (-1)^b q^{4b^2} y^{2b} \theta_1(h + 2\pi i b \tau)$, $\theta_1(h + 2\pi i a) = (-1)^a \theta_1(z)$,
- (3) $\theta_4(h) = e^{\pi i(b-\frac{1}{2})} q^{(2b+1)^2} y^{2b+1} \theta_1(h + 2\pi i(b + \frac{1}{2})\tau)$,
- (4) $\theta_1(h) = e^{\pi i(b-\frac{1}{2})} q^{(2b+1)^2} y^{2b+1} \theta_4(h + 2\pi i(b + \frac{1}{2})\tau)$.

Proof. All these formulas follow by straightforward induction from (4.6) and (4.7). As an illustration we check (1) and (3). The formula $\theta_4(h + 2\pi i \tau) = -q^{-4} y^{-2} \theta_4(h)$ gives by induction

$$\begin{aligned} \theta_4(h + 2\pi i b \tau) &= -q^{-4} e^{-(h+2\pi i(b-1)\tau)} \theta_4(h + 2\pi i(b-1)\tau) \\ &= -q^{-8b+4} y^{-2} (-1)^{-(b-1)} q^{-4(b-1)^2} y^{-(2b-2)} \theta_4(h) = (-1)^{-b} q^{-(2b)^2} y^{-2b} \theta_4(h), \end{aligned}$$

and (1) follows. Similarly

$$\begin{aligned} \theta_4(h + 2\pi i(b + 1/2)\tau) &= i q^{-1} e^{-h/2 - \pi i b \tau} \theta_1(h + 2\pi i b \tau) = i q^{-4b-1} y^{-1} (-1)^{-b} q^{-(2b)^2} y^{-2b} \theta_1(h) \\ &= e^{-\pi i(b-\frac{1}{2})} q^{-(2b+1)^2} y^{-(2b+1)} \theta_1(h), \end{aligned}$$

and (3) follows. □

4.2. Wallcrossing formula. Now we review the wallcrossing formula from [9], [10], and generalize it slightly. Let $\sigma(X)$ be the signature of X .

Definition 4.10. Let $r \geq 0$, let $\xi \in H^2(X, \mathbb{Z})$ with $\xi^2 < 0$. Let L be a line bundle on X . We put

$$\Delta_\xi^X(L, P^r) := 2i^{\langle \xi, K_X \rangle} \Lambda^2 q^{-\xi^2} y^{\langle \xi, (L-K_X) \rangle} \tilde{\theta}_4(h)^{(L-K_X)^2} \theta_4^{\sigma(X)} u' h^* M^r,$$

and put $\Delta_\xi^X(L) := \Delta_\xi^X(L, P^0)$. By the results of the previous section it can be developed as a power series

$$\Delta_\xi^X(L, P^r) = \sum_{d \geq 0} f_d(\tau) \Lambda^d \in \mathbb{C}((q))[[\Lambda]],$$

whose coefficients $f_d(\tau)$ are Laurent series in q . If $\langle \xi, L \rangle \equiv r \pmod{2}$, the *wallcrossing term* is defined as

$$\delta_\xi^X(L, P^r) := \sum_{d \geq 0} \delta_{\xi,d}^X(L, P^r) \Lambda^d \in \mathbb{Q}[[\Lambda]],$$

with

$$\delta_{\xi,d}^X(L, P^r) = \text{Coeff}_{q^0}[f_d(\tau)].$$

Again we write $\delta_{\xi,d}^X(L) := \delta_{\xi,d}^X(L, P^0)$ and $\delta_\xi^X(L) := \delta_\xi^X(L, P^0)$.

The wallcrossing terms $\delta_\xi^X(L) := \delta_{\xi,d}^X(L, P^0)$ were already introduced in [9] and used in [10]. As we will recall in a moment, they compute the change of the K-theoretic Donaldson invariants $\chi_{c_1}^{X,\omega}(L)$, when ω crosses a wall. Later we will introduce K-theoretic Donaldson invariants with point class $\chi_{c_1}^{X,\omega}(L, P^r)$, whose wallcrossing is computed by $\delta_\xi^X(L, P^r)$. Intuitively we want to think of r as the power of a K-theoretic point class \mathcal{P} .

Remark 4.11. (1) $\delta_{\xi,d}^X(L, P^r) = 0$ unless $d \equiv -\xi^2 \pmod{4}$.

(2) In the definition of $\delta_\xi^X(L, P^r)$ we can replace $\Delta_\xi^X(L, P^r)$ by

$$(4.12) \quad \begin{aligned} \overline{\Delta}_\xi^X(L, P^r) &:= \frac{1}{2}(\Delta_\xi^X(L, P^r) - \Delta_{-\xi}^X(L, P^r)) \\ &= M^r i^{\langle \xi, K_X \rangle} \Lambda^2 q^{-\xi^2} (y^{\langle \xi, (L-K_X) \rangle} - (-1)^{\xi^2} y^{-\langle \xi, (L-K_X) \rangle}) \widetilde{\theta}_4(h)^{(L-K_X)^2} \theta_4^\sigma(X) u' h^*. \end{aligned}$$

Proof. (1) As $h \in \mathbb{C}[[q^{-1}\Lambda, q^4]]$, we also have $h^*, y, \widetilde{\theta}_4(h), M \in \mathbb{C}[[q^{-1}\Lambda, q^4]]$. Finally $u, u' \in q^{-2}\mathbb{Q}[[q^4]]$. It follows that $\Delta_\xi^X(L, P^r) \in q^{-\xi^2}\mathbb{C}[[q^{-1}\Lambda, q^4]]$. Writing $\Delta_\xi^X(L, P^r) = \sum_d f_{d,r}(\tau)\Lambda^d$, we see that $\text{Coeff}_{q^0}[f_{d,r}(\tau)] = 0$ unless $d \equiv -\xi^2 \pmod{4}$. (2) Note that $\widetilde{\theta}_4(h)$ is even in Λ and h^* is odd in Λ , thus $\overline{\Delta}_\xi^X(L, P^r) = \sum_{d \equiv -\xi^2(2)} f_{d,r}(\tau)\Lambda^d$, and the claim follows by (1). \square

The main result of [9] is the following (see also [10]).

Theorem 4.13. *Let H_1, H_2 be ample divisors on X , assume that $\langle H_1, K_X \rangle < 0, \langle H_2, K_X \rangle < 0$, and that H_1, H_2 do not lie on a wall of type (c_1, d) . Then*

$$\chi(M_{H_1}^X(c_1, d), \mu(L)) - \chi(M_{H_2}^X(c_1, d), \mu(L)) = \sum_{\xi} \delta_{\xi, d}^X(L),$$

where ξ runs through all classes of type (c_1, d) with $\langle \xi, H_1 \rangle > 0 > \langle \xi, H_2 \rangle$.

Note that the condition $\langle H_1, K_X \rangle < 0, \langle H_2, K_X \rangle < 0$ implies that all the classes of type (c_1, d) with $\langle \xi, H_1 \rangle > 0 > \langle \xi, H_2 \rangle$ are good in the sense of [9], so the wallcrossing formula there applies. Let $c_1 \in H^2(X, \mathbb{Z})$. Let H_1, H_2 be ample on X , assume they do not lie on a wall of type (c_1) . Then it follows that

$$\chi_{c_1}^{X, H_1}(L) - \chi_{c_1}^{X, H_2}(L) = \sum_{\xi} \delta_{\xi}^X(L),$$

where ξ runs through all classes in $c_1 + 2H^2(X, \mathbb{Z})$ with $\langle \xi, H_1 \rangle > 0 > \langle \xi, H_2 \rangle$.

4.3. Polynomiality and vanishing of the wallcrossing. By definition the wallcrossing terms $\delta_\xi^X(L, P^r)$ are power series in Λ . We now show that they are always polynomials, modifying the proof of [10, Thm. 3.19]. We have seen above that $h \in iq^{-1}\Lambda\mathbb{Q}[[q^{-2}\Lambda^2, q^4]]$, and thus $y = e^{h/2} \in \mathbb{Q}[[iq^{-1}\Lambda, q^4]]$.

Lemma 4.14. ([10, Lem. 3.18])

(1) $\sinh(h/2) = y - y^{-1} \in iq^{-1}\Lambda\mathcal{R}, \frac{1}{\sinh(h/2)} \in iq\Lambda^{-1}\mathcal{R}$.

(2) For all integers n we have

$$\begin{aligned} \sinh((2n+1)h/2) &\in i\mathbb{Q}[q^{-1}\Lambda]_{|2n+1|}\mathcal{R}, \quad \cosh(nh) \in \mathbb{Q}[q^{-2}\Lambda^2]_{|n|}\mathcal{R}, \\ \sinh(nh)h^* &\in \mathbb{Q}[q^{-2}\Lambda^2]_{|n|}\mathcal{R} \quad \cosh((2n+1)h/2)h^* \in i\mathbb{Q}[q^{-1}\Lambda]_{|2n+1|}\mathcal{R}. \end{aligned}$$

(3) $\widetilde{\theta}_4(h) \in \mathcal{R}$, with $\widetilde{\theta}_4(h) = 1 + q^2\Lambda^2 + O(q^4)$.

Lemma 4.15. *Let $r \in \mathbb{Z}_{\geq 0}$, let $\xi \in H^2(X, \mathbb{Z})$, and L a line bundle on X with $\xi^2 < 0$ and $\langle \xi, L \rangle \equiv r \pmod{2}$.*

- (1) $\delta_{\xi,d}^X(L, P^r) = 0$ unless $-\xi^2 \leq d \leq \xi^2 + 2|\langle \xi, L - K_X \rangle| + 2r + 4$. In particular $\delta_{\xi}^X(L, P^r) \in \mathbb{Q}[\Lambda]$.
- (2) $\delta_{\xi}^X(L, P^r) = 0$ unless $-\xi^2 \leq |\langle \xi, L - K_X \rangle| + r + 2$. (Recall that by definition $\xi^2 < 0$).

Proof. Assume first that $r = 2l$ is even. Let $N := \langle \xi, L - K_X \rangle$. Then it is shown in the proof of [10, Thm. 3.19] that $\overline{\Delta}_{\xi}^X(L) = q^{-\xi^2} \mathbb{Q}[q^{-1}\Lambda]_{\leq |N|+2} \mathcal{R}$. On the other hand we note that $M^2 = 4(1 + u\Lambda^2 + \Lambda^4) \in \mathbb{Q}[q^{-2}\Lambda^2]_{\leq 1} \mathcal{R}$. Putting this together we get

$$\Delta_{\xi}^X(L, P^r) \in q^{-\xi^2} \mathbb{Q}[q^{-1}\Lambda]_{\leq |N|+r+2} \mathcal{R}.$$

Now assume that $r = 2l + 1$ is odd. If N is even, then by the condition that $\langle L, \xi \rangle$ is odd, we get $(-1)^{\xi^2} = -(-1)^N$, and therefore

$$\overline{\Delta}_{\xi}^X(L, P^r) = q^{-\xi^2} M^r i^{\langle \xi, K_X \rangle} \Lambda^2 \cosh(Nh/2) h^* \tilde{\theta}_4^{(L-K_X)^2} \theta_4^{\sigma(X)} u'.$$

By (4.5) we get $h^* M = \frac{4i\Lambda}{\theta_2\theta_3} \in i\Lambda q^{-1} \mathbb{Q}[q^4]$. Thus by Lemma 4.14 we get $\cosh(Nh/2) h^* M \in i\mathbb{Q}[q^{-1}\Lambda]_{\leq |N|+1} \mathcal{R}$. Using also that $\langle \xi, K_X \rangle \equiv \xi^2 \equiv 1 \pmod{2}$, that $M^2 \in \mathbb{Q}[q^{-2}\Lambda^2]_{\leq 1} \mathcal{R}$ and $\Lambda^2 u' \in q^{-2}\Lambda^2 \mathcal{R}$, we get again $\overline{\Delta}_{\xi}^X(L, P^r) \in q^{-\xi^2} \mathbb{Q}[q^{-1}\Lambda]_{\leq |N|+r+2} \mathcal{R}$. Finally, if N is odd, a similar argument shows that

$$\overline{\Delta}_{\xi}^X(L, P^r) = q^{-\xi^2} M^r i^{\langle \xi, K_X \rangle} \Lambda^2 \sinh(Nh/2) h^* \tilde{\theta}_4^{(L-K_X)^2} \theta_4^{\sigma(X)} u' \in q^{-\xi^2} \mathbb{Q}[q^{-1}\Lambda]_{\leq |N|+r+2} \mathcal{R}.$$

Therefore we have in all cases that $\delta_{\xi,d}^X(L, P^r) = 0$ unless $-\xi^2 - \min(d, 2|N| + 2r + 4 - d) \leq 0$, i.e. unless $-\xi^2 \leq d \leq \xi^2 + 2|N| + 2r + 4$. In particular $\delta_{\xi}^X(L, P^r) = 0$ unless $-\xi^2 \leq \xi^2 + 2|N| + 2r + 4$, i.e. unless $-\xi^2 \leq |N| + r + 2$. \square

Remark 4.16. We note that this implies that for ξ a class of type (c_1) , $\delta_{\xi,d}^X(L) = 0$ for all L unless ξ a class of type (c_1, d) .

5. INDEFINITE THETA FUNCTIONS, VANISHING, INVARIANTS WITH POINT CLASS

We want to study the K -theoretic Donaldson invariants for polarizations on the boundary of the ample cone. Let $F \in H^2(X, \mathbb{Z})$ the class of an effective divisor with $F^2 = 0$ and such that F is nef, i.e. $\langle F, C \rangle \geq 0$ for any effective curve in X . Then F is a limit of ample classes. Let $c_1 \in H^2(X, \mathbb{Z})$ such that $\langle c_1, F \rangle$ is odd. Fix $d \in \mathbb{Z}$ with $d \equiv -c_1^2 \pmod{4}$. Let ω be ample on X . Then for $n > 0$ sufficiently large $nF + \omega$ is ample on X and there is no wall ξ of type (c_1, d) with $\langle \xi, (nF + \omega) \rangle > 0 > \langle \xi, F \rangle$. Let $L \in \text{Pic}(X)$ and $r \in \mathbb{Z}_{\geq 0}$ with $\langle c_1, L \rangle$ even. Thus we define for n sufficiently large

$$\begin{aligned} M_F^X(c_1, d) &:= M_{nF+\omega}^X(c_1, d), \\ \chi(M_F^X(c_1, d), \mu(L)) &:= \chi(M_{nF+\omega}^X(c_1, d), \mu(L)), \\ \chi_{c_1}^{X,F}(L) &:= \sum_{d \geq 0} \chi(M_F^X(c_1, d), \mu(L)) \Lambda^d. \end{aligned}$$

We use the following standard fact.

Remark 5.1. Let X be a simply connected algebraic surface, and let $\pi : X \rightarrow \mathbb{P}^1$ be a morphism whose general fibre is isomorphic to \mathbb{P}^1 . Let $F \in H^2(X, \mathbb{Z})$ be the class of a fibre. Then F is nef. Assume that $\langle c_1, F \rangle$ is odd. Then $M_F^X(c_1, d) = \emptyset$ for all d . Thus $\chi(M_F^X(c_1, d), \mu(L)) = 0$ for all $d \geq 0$. Thus if ω ample on X and does not lie on a wall of type (c_1) , then

$$\chi_{c_1}^{X, \omega}(L) = \sum_{\omega\xi > 0 > \xi F} \delta_\xi^X(L),$$

where the sum is over all classes ξ of type (c_1) with $\omega\xi > 0 > \xi F$.

5.1. Theta functions for indefinite lattices. We briefly review a few facts about theta functions for indefinite lattices of type $(r-1, 1)$ introduced in [11]. More can be found in [11], [10]. For us a *lattice* is a free \mathbb{Z} -module Γ together with a quadratic form $Q : \Gamma \rightarrow \frac{1}{2}\mathbb{Z}$, such that the associated bilinear form $x \cdot y := Q(x+y) - Q(x) - Q(y)$ is nondegenerate and \mathbb{Z} -valued. We denote the extension of the quadratic and bilinear form to $\Gamma_{\mathbb{R}} := \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ and $\Gamma_{\mathbb{C}} := \Gamma \otimes_{\mathbb{Z}} \mathbb{C}$ by the same letters. We will consider the case that Γ is $H^2(X, \mathbb{Z})$ for a rational surface X with the *negative* of the intersection form. Thus for $\alpha, \beta \in H^2(X, \mathbb{Z})$ we have $Q(\alpha) = -\frac{\alpha^2}{2}$, $\alpha \cdot \beta = -\langle \alpha, \beta \rangle$. Now let Γ be a lattice of rank r . Denote by M_Γ the set of meromorphic maps $f : \Gamma_{\mathbb{C}} \times \mathcal{H} \rightarrow \mathbb{C}$. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2, \mathbb{Z})$, we define a map $|_k A : M_\Gamma \rightarrow M_\Gamma$ by

$$f|_k A(x, \tau) := (c\tau + d)^{-k} \exp\left(-2\pi i \frac{cQ(x)}{c\tau + d}\right) f\left(\frac{x}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right).$$

Then $|_k A$ defines an action of $Sl(2, \mathbb{Z})$ on M_Γ . We denote

$$S_\Gamma := \{f \in \Gamma \mid f \text{ primitive}, Q(f) = 0, f \cdot h < 0\}, \quad C_\Gamma := \{m \in \Gamma_{\mathbb{R}} \mid Q(m) < 0, m \cdot h < 0\}.$$

For $f \in S_\Gamma$ put $D(f) := \{(\tau, x) \in \mathcal{H} \times \Gamma_{\mathbb{C}} \mid 0 < \Im(f \cdot x) < \Im(\tau)/2\}$, and for $h \in C_\Gamma$ put $D(h) = \mathcal{H} \times \Gamma_{\mathbb{C}}$. For $t \in \mathbb{R}$ denote

$$\mu(t) := \begin{cases} 1 & t \geq 0, \\ 0 & t < 0. \end{cases}$$

Let $c, b \in \Gamma$. Let $f, g \in S_\Gamma \cup C_\Gamma$. Then for $(\tau, x) \in D(f) \cap D(g)$ define

$$\Theta_{\Gamma, c, b}^{f, g}(\tau, x) := \sum_{\xi \in \Gamma + c/2} (\mu(\xi \cdot f) - \mu(\xi \cdot g)) e^{2\pi i \tau Q(\xi)} e^{2\pi i \xi \cdot (x + b/2)}.$$

$$\text{Let } T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in Sl(2, \mathbb{Z}).$$

Theorem 5.2. (1) For $f, g \in S_\Gamma$ the function $\Theta_{X, c, b}^{f, g}(\tau, x)$ has an meromorphic continuation to $\mathcal{H} \times \Gamma_{\mathbb{C}}$.

(2) For $|\Im(f \cdot x)/\Im(\tau)| < 1/2$ and $|\Im(g \cdot x)/\Im(\tau)| < 1/2$ it has a Fourier development

$$\Theta_{X,c,b}^{f,g}(x, \tau) := \frac{1}{1 - e^{2\pi i f \cdot (x+b/2)}} \sum_{\substack{\xi \cdot f = 0 \\ f \cdot g \leq \xi \cdot g < 0}} e^{2\pi i \tau Q(\xi)} e^{2\pi i \xi \cdot (x+b/2)} \\ - \frac{1}{1 - e^{2\pi i g \cdot (x+b/2)}} \sum_{\substack{\xi \cdot g = 0 \\ f \cdot g \leq \xi \cdot f < 0}} e^{2\pi i \tau Q(\xi)} e^{2\pi i \xi \cdot (x+b/2)} + \sum_{\xi f > 0 > \xi g} e^{2\pi i \tau Q(\xi)} (e^{2\pi i \xi \cdot (x+b/2)} - e^{-2\pi i \xi \cdot (x+b/2)}),$$

where the sums are always over $\xi \in \Gamma + c/2$.

(3)

$$\begin{aligned} (\Theta_{X,c,b}^{f,g} \theta_3^{\sigma(\Gamma)})|_1 S &= (-1)^{-b \cdot c/2} \Theta_{X,b,c}^{f,g} \theta_3^{\sigma(\Gamma)}, \\ (\Theta_{X,c,b}^{f,g} \theta_3^{\sigma(\Gamma)})|_1 T &= (-1)^{3Q(c)/2 - cw/2} \Theta_{X,c,b-c+w}^{f,g} \theta_4^{\sigma(\Gamma)}, \\ (\Theta_{X,c,b}^{f,g} \theta_3^{\sigma(\Gamma)})|_1 T^2 &= (-1)^{-Q(c)} \Theta_{X,c,b}^{f,g} \theta_3^{\sigma(\Gamma)}, \\ (\Theta_{X,c,b}^{f,g} \theta_3^{\sigma(\Gamma)})|_1 T^{-1} S &= (-1)^{-Q(c)/2 - c \cdot b/2} \Theta_{X,w-c+b,c}^{f,g} \theta_2^{\sigma(\Gamma)}, \end{aligned}$$

where w is a characteristic element of Γ .

Remark 5.3. For $f, g, h \in C_\Gamma \cap S_\Gamma$ we have the cocycle condition. $\Theta_{\Gamma,c,b}^{f,g}(\tau, x) + \Theta_{\Gamma,c,b}^{g,h}(\tau, x) = \Theta_{\Gamma,c,b}^{f,h}(\tau, x)$, which holds wherever all three terms are defined.

In the following let X be a rational algebraic surface. We can express the difference of the K -theoretic Donaldson invariants for two different polarisations in terms of these indefinite theta functions. Here we take Γ to be $H^2(X, \mathbb{Z})$ with the *negative* of the intersection form, and we choose K_X as the characteristic element in Theorem 5.2(3).

Definition 5.4. Let $F, G \in S_\Gamma \cup C_\Gamma$, let $c_1 \in H^2(X, \mathbb{Z})$. We put

$$\Psi_{X,c_1}^{F,G}(L; \Lambda, \tau) := \Theta_{X,c_1,K_X}^{F,G} \left(\frac{(L - K_X)h}{2\pi i}, \tau \right) \Lambda^2 \tilde{\theta}_4(h)^{(L-K_X)^2} \theta_4^{\sigma(X)} u' h^*.$$

Lemma 5.5. Let H_1, H_2 be ample on X with $\langle H_1, K_X \rangle < 0$ and $\langle H_2, K_X \rangle < 0$, and assume that they do not lie on a wall of type (c_1) . Then

(1)

$$\Psi_{X,c_1}^{H_2,H_1}(L; \Lambda, \tau) M^r = \sum_{\xi} \overline{\Delta}_{\xi}^X(L, P^r),$$

where ξ runs through all classes on X of type (c_1) with $\langle H_2, \xi \rangle > 0 > \langle H_1, \xi \rangle$.

(2) $\chi_{c_1}^{X,H_2}(L) - \chi_{c_1}^{X,H_1}(L) = \text{Coeff}_{q^0} [\Psi_{X,c_1}^{H_2,H_1}(L; \Lambda, \tau)]$.

Proof. (2) is proven in [10, Cor. 4.6], where the assumption is made that $-K_X$ is ample, but the proof only uses $\langle H_1, K_X \rangle < 0$ and $\langle H_2, K_X \rangle < 0$, because this condition is sufficient for Theorem 4.13. The argument of [10, Cor. 4.6] actually shows (1) in case $r = 0$, but as $\overline{\Delta}_{\xi}^X(L, P^r) = \overline{\Delta}_{\xi}^X(L, P^0) M^r$, the case of general r follows immediately. \square

Following [10] we use Lemma 5.5 to extend the generating function $\chi_{c_1}^{X,\omega}(L)$ to $\omega \in S_L \cup C_L$.

Definition 5.6. Let η be ample on X with $\langle \eta, K_X \rangle < 0$, and not on a wall of type (c_1) . Let $\omega \in S_X \cup C_X$. We put

$$\chi_{c_1}^{X,\omega}(L) := \chi_{c_1}^{X,\eta}(L) + \text{Coeff}_{q^0} [\Psi_{X,c_1}^{\omega,\eta}(L; \Lambda, \tau)].$$

By the cocycle condition the definition of $\chi_{c_1}^{X,\omega}(L)$ is independent of the choice of η . Furthermore by Corollary 5.5 this coincides with the previous definition in case ω is also ample, $\langle \omega, K_X \rangle < 0$ and ω does not lie on a wall of type (c_1) . However if $\langle \omega, K_X \rangle \geq 0$, it is very well possible that the coefficient of Λ^d of $\chi_{c_1}^{X,\omega}(L)$ is different from $\chi(M_\omega^X(c_1, d), \mu(L))$.

Remark 5.7. Now let $H_1, H_2 \in S_X \cup C_X$. By the cocycle condition, we have

$$\chi_{c_1}^{X,H_2}(L) - \chi_{c_1}^{X,H_1}(L) = \text{Coeff}_{q^0} [\Psi_{X,c_1}^{H_2,H_1}(L; \Lambda, \tau)].$$

Proposition 5.8. Let X be a rational surface. Let $\omega \in C_X \cup S_X$. Let $c_1 \in H^2(X, \mathbb{Z})$. Let \hat{X} be the blowup of X in a general point, and E the exceptional divisor. Let $L \in \text{Pic}(X)$ with $\langle L, c_1 \rangle$ even. Then

- (1) $\chi_{c_1}^{\hat{X},\omega}(L) = \chi_{c_1}^{X,\omega}(L)$,
- (2) $\chi_{c_1+E}^{\hat{X},\omega}(L) = \Lambda \chi_{c_1}^{X,\omega}(L)$.

Proof. This is [10, Prop. 4.9], where the additional assumption is made that $-K_{\hat{X}}$ is ample. The proof works without this assumption with very minor modifications. In the original proof the result is first proven for an $H_0 \in C_X$ which does not lie on any wall of type (c_1) . We now have to assume in addition that $\langle H_0, K_X \rangle < 0$. The rest of the proof is unchanged. \square

In [10, Thm. 4.21] it is shown that if X is a rational surface with $-K_X$ ample, then $\chi_{c_1}^{X,F}(L) = \chi_{c_1}^{X,G}(L)$ for all $F, G \in S_X$. A modification of this proof shows the following.

Proposition 5.9. Let X be $\mathbb{P}^1 \times \mathbb{P}^1$ or a blowup of \mathbb{P}^2 in finitely many points. Let $L \in \text{Pic}(X)$, let $c_1 \in H^2(X, \mathbb{Z})$ with $\langle c_1, L \rangle$ even. Let $F, G \in S_X$. Assume that for all $W \in K_X + 2H^2(X, \mathbb{Z})$ with $\langle F, W \rangle \leq 0 \leq \langle G, W \rangle$, we have $W^2 < K_X^2$. Then $\chi_{c_1}^{X,F}(L) = \chi_{c_1}^{X,G}(L)$.

Proof. We know that $\chi_{c_1}^{X,F}(L) - \chi_{c_1}^{X,G}(L) = \text{Coeff}_{q^0} [\Psi_{X,c_1}^{F,G}(L, \Lambda, \tau)]$, and in the proof of [10, Thm. 4.21] it is shown that

$$\text{Coeff}_{q^0} [\Psi_{X,c_1}^{F,G}(L, \Lambda, \tau)] = -\frac{1}{4} \text{Coeff}_{q^0} [\tau^{-2} \Psi_{X,c_1}^{F,G}(L, \Lambda, S\tau)] - \frac{1}{4} i^{c_1^2+3} \text{Coeff}_{q^0} [\tau^{-2} \Psi_{X,c_1}^{F,G}(L, i\Lambda, S\tau)].$$

Therefore it is enough to show that $\text{Coeff}_{q^0} [\tau^{-2} \Psi_{X,c_1}^{F,G}(L, \Lambda, S\tau)] = 0$. Furthermore in the proof of [10, Thm. 4.21] we have seen that the three functions $\tilde{u} := -\frac{\theta_3^4 + \theta_4^4}{\theta_3^2 \theta_4^2}$,

$$\tilde{h} = -\frac{2}{\theta_4 \theta_3} \cdot \sum_{\substack{n \geq 0 \\ n \geq k \geq 0}} \binom{-\frac{1}{2}}{n} \binom{n}{k} \frac{\tilde{u}^k \Lambda^{4n-2k+1}}{4n-2k+1}, \quad \tilde{G}(\Lambda, \tau) = \frac{(-1)^{\langle c_1, K_X \rangle / 2 - \sigma(X)/4} \Lambda^3}{\theta_3^3 \theta_4^3 (1 + \tilde{u} \Lambda^2 + \Lambda^4)} \tilde{\theta}_2(\tilde{h})^{(L-K_X)^2}$$

are regular at $q = 0$, and furthermore that we can write

$$\tau^{-2} \Psi_{X, c_1}^{F, G}(L, \Lambda, S\tau) = \Theta_{X, K_X, c_1}^{F, G} \left(\frac{(L - K_X)\tilde{h}}{2\pi i}, \tau \right) \theta_2^{K_X^2} \tilde{G}(\Lambda, \tau).$$

(note that $\sigma(X) + 8 = K_X^2$ to compare with the formulas in the proof of [10, Thm. 4.19]). As $\theta_2^{K_X^2}$ starts with $q^{K_X^2}$, specializing the formula of Theorem 5.2(2) to the case $c = K_X$, $b = c_1$, $F = f$, $G = g$, we see that all the summands in $\Theta_{X, K_X, c_1}^{F, G} \left(\frac{(L - K_X)\tilde{h}}{2\pi i}, \tau \right)$ are of the form $q^{-W^2} J_W(\Lambda, \tau)$, where $J_W(\Lambda, \tau)$ is regular at $q = 0$ and $W \in K_X + 2H^2(X, \mathbb{Z})$ with $\langle F, W \rangle \leq 0 \leq \langle G, W \rangle$. The claim follows. \square

Corollary 5.10. *Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, or let X be the blowup of \mathbb{P}^2 in finitely many general points p_1, \dots, p_n with exceptional divisors E_1, \dots, E_n . In case $X = \mathbb{P}^1 \times \mathbb{P}^1$ let F be the class of a fibre of the projection to one of the two factors; otherwise let $F = H - E_i$ for some $i \in \{1, \dots, n\}$. Let $c_1 \in H^2(X, \mathbb{Z})$ and let L be a line bundle on X with $\langle L, c_1 \rangle$ even. Then*

$$(1) \chi_{c_1}^{X, F}(L) = 0.$$

(2) *Thus for all $\omega \in S_X \cup C_X$ we have*

$$\chi_{c_1}^{X, \omega}(L) = \text{Coeff}_{q^0} [\Psi_{X, c_1}^{\omega, F}(L; \Lambda, \tau)].$$

Proof. (1) Let \hat{X} be the blowup of X in a general point with exceptional divisor E . Then \hat{X} is the blowup of \mathbb{P}^2 in $n + 1$ general points, (with $n = 1$ in case $X = \mathbb{P}^1 \times \mathbb{P}^1$). We denote E_1, \dots, E_{n+1} the exceptional divisors, then we can assume that $F = H - E_1$. We put $G = H - E_{n+1}$. If $\langle c_1, H \rangle$ is even, we put $\hat{c}_1 = c_1 + E_{n+1}$, and if $\langle c_1, H \rangle$ is odd, we put $\hat{c}_1 = c_1$. Thus $\langle \hat{c}_1, G \rangle$ is odd and therefore by Remark 5.1 we get $\chi_{\hat{c}_1}^{\hat{X}, G}(L) = 0$. By Proposition 5.8 we have $\chi_{c_1}^{X, F}(L) = \chi_{\hat{c}_1}^{\hat{X}, F}(L)$ or $\chi_{c_1}^{X, F}(L) = \frac{1}{\Lambda} \chi_{\hat{c}_1}^{\hat{X}, F}(L)$. Therefore it is enough to show that $\chi_{\hat{c}_1}^{\hat{X}, F}(L) = \chi_{\hat{c}_1}^{\hat{X}, G}(L)$. So by Proposition 5.9 we need to show that for all $W \in K_{\hat{X}} + 2H^2(\hat{X}, \mathbb{Z})$ with $\langle F, W \rangle \leq 0 \leq \langle G, W \rangle$, we have $W^2 < K_{\hat{X}}^2$. Let $W = kH + a_1E_1 + \dots + a_{n+1}E_{n+1} \in K_{\hat{X}} + 2H^2(\hat{X}, \mathbb{Z})$ with $\langle F, W \rangle \leq 0 \leq \langle G, W \rangle$. Then k, a_1, \dots, a_{n+1} are odd integers, the condition $\langle F, W \rangle \leq 0$ gives that $k \leq a_1$, and the condition $\langle G, W \rangle \geq 0$ gives that $k \geq a_{n+1}$. So either $k < 0$ and $|a_1| \geq |k|$ or $k > 0$, and $|a_{n+1}| \geq |k|$. As all the a_i are odd, this gives

$$W^2 = k^2 - a_1^2 - \dots - a_{n+1}^2 \leq -n < 8 - n = K_X^2.$$

\square

5.2. Invariants with point class. We can now define K -theoretic Donaldson invariants with powers of the point class.

Corollary 5.11. *Let X be the blowup of \mathbb{P}^2 in general points p_1, \dots, p_r , with exceptional divisors E_1, \dots, E_r . Let \overline{X} be the blowup of \mathbb{P}^2 in general points q_1, \dots, q_r , with exceptional divisors $\overline{E}_1, \dots, \overline{E}_r$. For a class $M = dH + a_1E_1 + \dots + a_rE_r \in H^2(X, \mathbb{R})$ let $\overline{M} := dH + a_1\overline{E}_1 + \dots + a_r\overline{E}_r \in H^2(\overline{X}, \mathbb{R})$. Then for all $L \in \text{Pic}(X)$, $c_1 \in H^2(X, \mathbb{Z})$ with $\langle L, c_1 \rangle$ even, $\omega \in C_X \cup S_X$, we have $\chi_{c_1}^{X, \omega}(L) = \chi_{\overline{c}_1}^{\overline{X}, \overline{\omega}}(\overline{L})$.*

Proof. Let $F = H - E_1 \in S_X$, then $\overline{F} = H - \overline{E}_1 \in S_{\overline{X}}$, and thus $\chi_{c_1}^{X,F}(L) = 0 = \chi_{\overline{c}_1}^{\overline{X},\overline{F}}(\overline{L})$. The map sending E_i to \overline{E}_i for all i is an isomorphism of lattices, thus $\Psi_{X,c_1}^{\omega,F}(L; \Lambda, \tau) = \Psi_{\overline{X},\overline{c}_1}^{\overline{\omega},\overline{F}}(\overline{L}; \Lambda, \tau)$. Thus we get by Corollary 5.10 that

$$\chi_{c_1}^{X,\omega}(L) = \text{Coeff}_{q^0} [\Psi_{X,c_1}^{\omega,F}(L; \Lambda, \tau)] = \text{Coeff}_{q^0} [\Psi_{\overline{X},\overline{c}_1}^{\overline{\omega},\overline{F}}(\overline{L}; \Lambda, \tau)] = \chi_{\overline{c}_1}^{\overline{X},\overline{\omega}}(\overline{L}).$$

□

Definition 5.12. Let X be $\mathbb{P}^1 \times \mathbb{P}^1$ or the blowup of \mathbb{P}^2 in finitely many general points. Let $\omega \in S_X \cup C_X$, $c_1 \in H^2(X, \mathbb{Z})$, $L \in \text{Pic}(X)$. Let X_r be the blowup of X in r general points, with exceptional divisors E_1, \dots, E_r . Write $E := E_1 + \dots + E_r$. We put

$$\chi_{c_1}^{X,\omega}(L, P^r) := \Lambda^{-r} \chi_{c_1+E}^{X,\omega}(L - E), \quad \chi_{c_1,d}^{X,\omega}(L, P^r) := \text{Coeff}_{\Lambda^d} [\chi_{c_1}^{X,\omega}(L, P^r)].$$

We call the $\chi_{c_1,d}^{X,\omega}(L, P^r)$, $\chi_{c_1}^{X,\omega}(L, P^r)$ the K -theoretic Donaldson invariants with point class. More generally, if $F(\Lambda, P) = \sum_{i,j} a_{i,j} \Lambda^i P^j \in \mathbb{Q}[\Lambda, P]$ is a polynomial, we put

$$\chi_{c_1}^{X,\omega}(L, F(\Lambda, P)) := \sum_{i,j} a_{i,j} \Lambda^i \chi_{c_1}^{X,\omega}(L, P^j).$$

Remark 5.13. There should be a K -theory class \mathcal{P} on $M_\omega^X(c_1, d)$, such that $\chi_{c_1,d}^{X,\omega}(L, P^r) = \chi(M_\omega^X(c_1, d), \mu(L) \otimes \mathcal{P}^r)$. By the definition $\chi_{c_1}^{X,M}(L, P) = \Lambda^{-r} \chi_{c_1+E}^{\widehat{X},M}(L - E)$ the sheaf \mathcal{P} would encode local information at the blown-up point. We could view \mathcal{P} as a K -theoretic analogue of the point class in Donaldson theory. This is our motivation for the name of $\chi_{c_1}^{X,\omega}(L, P^r)$. For the moment we do not attempt to give a definition of this class \mathcal{P} . There are already speculations about possible definitions of K -theoretic Donaldson invariants with powers of the point class in [9, Sect. 1.3], and the introduction of $\chi_{c_1}^{X,\omega}(L, P^r)$ is motivated by that, but for the moment we do not try to make a connection to the approach in [9].

6. BLOWUP POLYNOMIALS, BLOWUP FORMULAS AND BLOWDOWN FORMULAS

In [10, Section 4.6] the blowup polynomials $R_n(\lambda, x)$, $S_n(\lambda, x)$ are introduced. They play a central role in our approach. In this section we will first show that they express all the K -theoretic Donaldson invariants of the blowup \widehat{X} of a surface X in terms of the K -theoretic Donaldson invariants of X . On the other hand we will use them to show that a small part of the K -theoretic Donaldson invariants of the blowup \widehat{X} determine *all* the K -theoretic Donaldson invariants of X (and thus by the above all the K -theoretic Donaldson invariants of any blowup of X , including \widehat{X}). Finally (as already in [10] in some cases), in the next section, we will use the blowup polynomials to construct recursion relations for many K -theoretic Donaldson invariants of rational ruled surfaces, enough to apply the above-mentioned results and determine all K -theoretic Donaldson invariants of \mathbb{P}^2 and thus of any blowup of \mathbb{P}^2 .

6.1. Blowup Polynomials and blowup formulas.

Definition 6.1. Define for all $n \in \mathbb{Z}$ rational functions $R_n, S_n \in \mathbb{Q}(\lambda, x)$ by $R_0 = R_1 = 1$, $S_1 = \lambda$, $S_2 = \lambda x$, the recursion relations

$$(6.2) \quad R_{n+1} = \frac{R_n^2 - \lambda^2 S_n^2}{R_{n-1}}, \quad n \geq 1; \quad S_{n+1} = \frac{S_n^2 - \lambda^2 R_n^2}{S_{n-1}}, \quad n \geq 2.$$

and $R_{-n} = R_n$, $S_{-n} = -S_n$. We will prove later that the R_n, S_n are indeed polynomials in λ, x .

The definition gives

$$\begin{aligned} R_1 &= 1, \quad R_2 = (1 - \lambda^4), \quad R_3 = -\lambda^4 x^2 + (1 - \lambda^4)^2, \quad R_4 = -\lambda^4 x^4 + (1 - \lambda^4)^4, \\ R_5 &= -\lambda^4 x^2 (x^4 + (2 - \lambda^4)(1 - \lambda^4)^2 x^2 + 3(1 - \lambda^4)^3) + (1 - \lambda^4)^6, \\ S_1 &= \lambda, \quad S_2 = \lambda x, \quad S_3 = \lambda(x^2 - (1 - \lambda^4)^2), \quad S_4 = \lambda x((1 - \lambda^8)x^2 - 2(1 - \lambda^4)^3). \end{aligned}$$

Proposition 6.3. ([10, Prop. 4.7])

$$(6.4) \quad \frac{\tilde{\theta}_4(nh)}{\theta_4(h)^{n^2}} = R_n(\Lambda, M), \quad \frac{\tilde{\theta}_1(nh)}{\theta_4(h)^{n^2}} = S_n(\Lambda, M).$$

In the following Proposition 6.5 and Corollary 6.6 let $X = \mathbb{P}^1 \times \mathbb{P}^1$, or let X be the blowup of \mathbb{P}^2 in finitely many general points p_1, \dots, p_n with exceptional divisors E_1, \dots, E_n . In case $X = \mathbb{P}^1 \times \mathbb{P}^1$ let F be the class of a fibre of the projection to one of the two factors; otherwise let $F = H - E_i$ for some $i \in \{1, \dots, n\}$.

Proposition 6.5. Let $c_1 \in H^2(X, \mathbb{Z})$ and let L be a line bundle on X with $\langle c_1, L \rangle$ even. Let \hat{X} be the blowup of X in a point, and let E be the exceptional divisor. Let $\omega \in C_X \cup S_X$. Then

- (1) $\chi_{c_1}^{\hat{X}, \omega}(L - (n-1)E) = \text{Coeff}_{q^0} [\Psi_{X, c_1}^{\omega, F}(L, \Lambda, \tau) R_n(\Lambda, M)]$.
- (2) $\chi_{c_1 + E}^{\hat{X}, \omega}(L - (n-1)E) = \text{Coeff}_{q^0} [\Psi_{X, c_1}^{\omega, F}(L, \Lambda, \tau) S_n(\Lambda, M)]$.

Proof. In [Prop. 4.34][10] it is proven for $X = \mathbb{P}^1 \times \mathbb{P}^1$ or the blowup of \mathbb{P}^2 in at most 7 points, and any $F, \omega \in C_X \cup S_X$ that

$$\begin{aligned} \Psi_{\hat{X}, c_1}^{\omega, F}(L - (n-1)E; \Lambda, \tau) &= \Psi_{X, c_1}^{\omega, F}(L, \Lambda, \tau) R_n(\Lambda, M) \\ \Psi_{\hat{X}, c_1 + E}^{\omega, F}(L - (n-1)E; \Lambda, \tau) &= \Psi_{X, c_1}^{\omega, F}(L, \Lambda, \tau) S_n(\Lambda, M), \end{aligned}$$

But the proof works without modification also for X the blowup of \mathbb{P}^2 in finitely many points. The result follows by Corollary 5.10. \square

We now see that the wall-crossing for the K -theoretic Donaldson invariants $\chi_{c_1}^{X, \omega}(L, P^r)$ with point class is given by the wallcrossing terms $\delta_\xi^X(L, P^r)$.

Corollary 6.6. (1) Let $r \geq 0$ and let L be a line bundle on X with $\langle L, c_1 \rangle \equiv r \pmod{2}$.

Then

$$\chi_{c_1}^{X, \omega}(L, P^r) = \text{Coeff}_{q^0} [\Psi_{X, c_1}^{\omega, F}(L, \Lambda, \tau) M^r].$$

(2) let $H_1, H_2 \in C_X$, not on a wall of type (c_1) . Then

$$\begin{aligned}\chi_{c_1}^{X, H_2}(L, P^r) - \chi_{c_1}^{X, H_2}(L, P^r) &= \sum_{\xi} \delta_{\xi}^X(L, P^r), \\ \chi_{c_1, d}^{X, H_2}(L, P^r) - \chi_{c_1, d}^{X, H_2}(L, P^r) &= \sum_{\xi} \delta_{\xi, d}^X(L, P^r),\end{aligned}$$

where in the first (resp. second) sum ξ runs through all classes of type (c_1) (resp. (c_1, d)) with $\langle H_1, \xi \rangle < 0 < \langle H_2, \xi \rangle$.

Proof. (1) Let \tilde{X} be the blowup of X in r general points and let $\overline{E} = \overline{E}_1 + \dots + \overline{E}_r$ be the sum of the exceptional divisors. Then by definition and iteration of Proposition 6.5 we have

$$\chi_{c_1}^{X, \omega}(L, P^r) = \frac{1}{\Lambda^r} \chi_{c_1 + \overline{E}}^{\tilde{X}, \omega}(L - \overline{E}) = \frac{1}{\Lambda^r} \text{Coeff}_{q^0} [\Psi_{X, c_1}^{\omega, F}(L, \Lambda, \tau) S_2(\Lambda, M)^r].$$

The claim follows because $S_2(\Lambda, M) = \Lambda M$. (2) By definition $\text{Coeff}_{q^0} [\overline{\Delta}_{\xi}^X(L, P^r)] = \delta_{\xi}^X(L, P^r)$, therefore by Lemma 5.5, and using also Remark 4.16, (2) follows from (1). \square

With this we get a general blowup formula.

Theorem 6.7. *Let X be \mathbb{P}^2 , $\mathbb{P}^1 \times \mathbb{P}^1$ or the blowup of \mathbb{P}^2 in finitely many general points. Let $c_1 \in H^2(X, \mathbb{Z})$ and let L be a line bundle on X and let $r \in \mathbb{Z}_{\geq 0}$ with $\langle c_1, L \rangle \equiv r \pmod{2}$. Let $\omega \in C_X \cup S_X$. Let \hat{X} be the blowup of X in a general point with exceptional divisor E . Then*

- (1) $\chi_{c_1}^{\hat{X}, \omega}(L - (n-1)E, P^r) = \chi_{c_1}^{X, \omega}(L, P^r \cdot R_n(\Lambda, P)),$
- (2) $\chi_{c_1 + E}^{\hat{X}, \omega}(L - (n-1)E, P^r) = \chi_{c_1}^{X, \omega}(L, P^r \cdot S_n(\Lambda, P)).$

Proof. If $X = \mathbb{P}^2$, then we apply Proposition 5.8 to reduce to the case that X is the blowup of \mathbb{P}^2 in a point. Thus we can by Corollary 5.10 and the definition of $\chi_{c_1}^{X, G}(L, P^s)$, assume that there is an $G \in S_X$ with $\chi_{c_1}^{X, G}(L, P^s) = 0$ for all $s \geq 0$.

(1) Let \tilde{X} be the blowup of X in r general points, with exceptional divisors F_1, \dots, F_r and put $F := F_1 + \dots + F_r$, and let \overline{X} the blowup of \tilde{X} in a point with exceptional divisor E . Then by definition

$$\chi_{c_1}^{\hat{X}, \omega}(L - (n-1)E, P^r) = \chi_{c_1 + F}^{\overline{X}, \omega}(L - F - (n-1)E).$$

We get by Corollary 5.10 that

$$\chi_{c_1 + F}^{\overline{X}, \omega}(L - F - (n-1)E) = \text{Coeff}_{q^0} [\Psi_{\tilde{X}, c_1 + F}^{\omega, G}(L - F, \Lambda, \tau) R_n(\Lambda, M)].$$

On the other hand, by Corollary 6.6 we get

$$\text{Coeff}_{q^0} [\Psi_{\tilde{X}, c_1 + F}^{\omega, G}(L - F, \Lambda, \tau) R_n(\Lambda, M)] = \chi_{c_1 + F}^{\tilde{X}, \omega}(L - F, R_n(\Lambda, P)) = \chi_{c_1}^{X, \omega}(L, P^r \cdot R_n(\Lambda, P)).$$

The proof of (2) is similar. \square

6.2. Further properties of the blowup polynomials.

Proposition 6.8. (1) For all $n \in \mathbb{Z}$, we have $R_n \in \mathbb{Z}[\lambda^4, x^2]$, $S_{2n+1} \in \lambda \mathbb{Z}[\lambda^4, x^2]$, $S_{2n} \in \lambda x \mathbb{Z}[\lambda^4, x^2]$.

(2) $R_n(\lambda, -x) = R_n(\lambda, x)$ and $S_n(\lambda, -x) = (-1)^{n-1} S_n(\lambda, x)$.

(3) The R_n , S_n satisfy the symmetries

$$\begin{aligned} R_{2n}\left(\frac{1}{\lambda}, \frac{x}{\lambda^2}\right) &= \frac{(-1)^n}{\lambda^{(2n)^2}} R_{2n}(\lambda, x), & S_{2n}\left(\frac{1}{\lambda}, \frac{x}{\lambda^2}\right) &= \frac{(-1)^{n-1}}{\lambda^{(2n)^2}} S_{2n}(\lambda, x), \\ R_{2n+1}\left(\frac{1}{\lambda}, \frac{x}{\lambda^2}\right) &= \frac{(-1)^n}{\lambda^{(2n+1)^2}} S_{2n+1}(\lambda, x), & S_{2n+1}\left(\frac{1}{\lambda}, \frac{x}{\lambda^2}\right) &= \frac{(-1)^n}{\lambda^{(2n+1)^2}} R_{2n+1}(\lambda, x). \end{aligned}$$

(4) For all $k, n \in \mathbb{Z}$, we have the relations

$$(6.9) \quad R_{2n} = R_n^4 - S_n^4, \quad S_{2n} = \frac{1}{\lambda} R_n S_n (S_{n+1} R_{n-1} - R_{n+1} S_{n-1}).$$

Proof. We write

$$\tilde{R}_n(h) := \frac{\tilde{\theta}_4(nh)}{\tilde{\theta}_4(h)^{n^2}} = R_n(\Lambda, M), \quad \tilde{S}_n(h) := \frac{\tilde{\theta}_1(nh)}{\tilde{\theta}_4(h)^{n^2}} = S_n(\Lambda, M),$$

where we have used (6.4). It is easy to see that Λ and M are algebraically independent, i.e. there exists no polynomial $f \in \mathbb{Q}[\lambda, x] \setminus \{0\}$, such that $f(\Lambda, M) = 0$ as a function on $\mathcal{H} \times \mathbb{C}$. For this, note that by $M^2 = 4(1 + u\Lambda^2 + \Lambda^4)$, the algebraic independence of Λ and M is equivalent to that of Λ and u . But this is clear, because as Laurent series in q , y , u is a Laurent series in q starting with $-\frac{1}{4q^{-2}}$ and Λ depends on y in a nontrivial way. As M and Λ are algebraically independent, R_n , S_n are the unique rational functions satisfying (6.4).

Now we will show (4). For any $k \in \mathbb{Z}$ we also have

$$(6.10) \quad \begin{aligned} \tilde{R}_{kn}(h) &= \frac{\tilde{\theta}_4(knh)}{\tilde{\theta}_4(h)^{k^2 n^2}} = \frac{\tilde{\theta}_4(knh)}{\tilde{\theta}_4(nh)^{k^2}} \left(\frac{\tilde{\theta}_4(nh)}{\tilde{\theta}_4(h)^{n^2}} \right)^{k^2} = \tilde{R}_k(nh) \tilde{R}_n(h)^{k^2}, \\ \tilde{S}_{kn}(h) &= \frac{\tilde{\theta}_1(knz)}{\tilde{\theta}_4(nh)^{k^2}} = \tilde{S}_k(nh) \tilde{R}_n(h)^{k^2} \end{aligned}$$

Thus, using $\tilde{R}_2(h) = 1 - \Lambda^4$, we find in particular

$$\tilde{R}_{2n}(h) = \tilde{R}_2(nh) \tilde{R}_n(h)^4 = \left(1 - \left(\frac{\tilde{\theta}_1(nh)}{\tilde{\theta}_4(nh)} \right)^4 \right) \tilde{R}_n(h)^4 = \tilde{R}_n(h)^4 - \tilde{S}_n(h)^4;$$

i.e., using the algebraic independence of Λ and M , $R_{2n} = R_n^4 - S_n^4$. In the same way we have

$$\tilde{S}_{2n}(h) = \tilde{S}_2(nh) \tilde{R}_n(h)^4 = \Lambda(nh) M(nh) \tilde{R}_n(h)^4.$$

By definition $\Lambda(nh) = \frac{\tilde{\theta}_1(nh)}{\tilde{\theta}_4(nh)} = \frac{\tilde{S}_n(h)}{\tilde{R}_n(h)}$. Now take the difference of the two formulas (see [22, §2.1 Ex. 3])

$$\theta_1(y \pm z) \theta_4(y \mp z) \theta_2 \theta_3 = \theta_1(y) \theta_4(y) \theta_2(z) \theta_3(z) \pm \theta_2(y) \theta_3(y) \theta_1(z) \theta_4(z)$$

with $y = nh$, $z = h$ to get

$$\theta_1((n+1)h)\theta_4((n-1)h)\theta_2\theta_3 - \theta_1((n-1)h)\theta_4((n+1)h)\theta_2\theta_3 = 2\theta_2(nh)\theta_3(nh)\theta_1(h)\theta_4(h).$$

This gives

$$\begin{aligned} M(nh) &= 2 \frac{\theta_2(nh)\theta_3(nh)}{\theta_2\theta_3\theta_4(nh)^2} = \frac{\theta_1((n+1)h)\theta_4((n-1)h) - \theta_1((n-1)h)\theta_4((n+1)h)}{\theta_1(h)\theta_4(h)\theta_4(nh)^2} \\ &= \frac{1}{\Lambda} \frac{\tilde{S}_{n+1}(h)\tilde{R}_{n-1}(h) - \tilde{S}_{n-1}(h)\tilde{R}_{n+1}(h)}{\tilde{R}_n(h)^2} \end{aligned}$$

Thus $S_{2n} = \frac{1}{\lambda} S_n R_n (S_{n+1} R_{n-1} - S_{n-1} R_{n+1})$. This shows (4)

(1) Next we will show that $R_n \in \mathbb{Z}[\lambda^4, x]$ and $S_n \in \lambda\mathbb{Z}[\lambda^4, x]$ for all $n \in \mathbb{Z}$. By symmetry it is enough to show this if n is a nonnegative integer. We know that this is true for $0 \leq n \leq 4$. Now assume that $m \geq 2$, and that we know the statement for all $0 \leq n \leq 2m$. Therefore $R_{m+1} \in \mathbb{Z}[\lambda^4, x]$, $S_{m+1} \in \lambda\mathbb{Z}[\lambda^4, x]$, and the formulas (6.9) give that $R_{2m+2} \in \mathbb{Z}[\lambda^4, x]$, $S_{2m+2} \in \lambda\mathbb{Z}[\lambda^4, x]$. The relations (6.2) say that

$$R_{2m+2}R_{2m} = R_{2m+1}^2 - \lambda^2 S_{2m+1}^2, \quad S_{2m+2}S_{2m} = S_{2m+1}^2 - \lambda^2 R_{2m+1}^2,$$

and thus

$$\begin{aligned} (6.11) \quad (1 - \lambda^4)R_{2m+1}^2 &= R_{2m+2}R_{2m} + \lambda^2 S_{2m+2}S_{2m}, \\ (1 - \lambda^4)S_{2m+1}^2 &= S_{2m+2}S_{2m} + \lambda^2 R_{2m+2}R_{2m}. \end{aligned}$$

Thus we get $(1 - \lambda^4)R_{2m+1}^2 \in \mathbb{Z}[\lambda^4, x]$ and $(1 - \lambda^4)S_{2m+1}^2 \in \lambda^2\mathbb{Z}[\lambda^4, x]$. Therefore, as $1 - \lambda^4$ is squarefree in $\mathbb{Q}[\lambda, x]$, we also have $R_{2m+1}^2 \in \mathbb{Z}[\lambda^4, x]$ and $S_{2m+1}^2 \in \lambda^2\mathbb{Z}[\lambda^4, x]$. As we already know that $R_{2m+1}, S_{2m+1} \in \mathbb{Q}(\lambda, x)$, this gives $R_{2m+1} \in \mathbb{Z}[\lambda^4, x]$ and $S_{2m+1} \in \lambda\mathbb{Z}[\lambda^4, x]$. So $R_{2m+1}, R_{2m+2} \in \mathbb{Z}[\lambda^4, x]$ and $S_{2m+1}, S_{2m+2} \in \lambda\mathbb{Z}[\lambda^4, x]$. Thus by induction on m , we get $R_n \in \mathbb{Z}[\lambda^4, x]$, $S_n \in \lambda\mathbb{Z}[\lambda^4, x]$.

(2) For $n = 0, 1, 2$ we see immediately that the R_n are even in x and the S_n have parity $(-1)^{n-1}$ in x . On the other hand the recursion formulas (6.2) say that R_{n+1} has the same parity as R_{n-1} in x and S_{n+1} the same parity as S_{n-1} . This also shows that $R_n \in \mathbb{Z}[\lambda^4, x^2]$, $S_{2n} \in \lambda x\mathbb{Z}[\lambda^4, x^2]$, $S_{2n+1} \in \lambda\mathbb{Z}[\lambda^4, x^2]$.

(3) The formulas (4.6), (4.7), (4.8) imply

$$(6.12) \quad \Lambda(h + \pi i\tau) = \frac{\theta_4(h)}{\theta_1(h)} = \frac{1}{\Lambda}, \quad M(h + \pi i\tau) = -2 \frac{\tilde{\theta}_2(h)\tilde{\theta}_3(h)}{\tilde{\theta}_1(h)^2} = -\frac{M}{\Lambda^2}.$$

Part (1) of Lemma 4.9 shows

$$\theta_4(2nh + 2\pi i n\tau) = (-1)^n q^{-4n^2} (y^{2n})^{-2n} \theta_4(2nh).$$

Thus, using (4.6) again, we get

$$\tilde{R}_{2n}(h + \pi i\tau) = (-1)^n \frac{\tilde{\theta}_4(2nh)}{\tilde{\theta}_1(h)^{4n^2}} = (-1)^n \frac{\tilde{R}_{2n}(h)}{\Lambda^{(2n)^2}}.$$

As Λ^4 and M are algebraically independent, (6.12) and (6.13) imply that

$$(6.13) \quad R_{2n}\left(\frac{1}{\lambda}, \frac{x}{\lambda^2}\right) = R_{2n}\left(\frac{1}{\lambda}, -\frac{x}{\lambda^2}\right) = (-1)^n \frac{1}{\lambda^{(2n)^2}} R_{2n}(\lambda, x).$$

The same argument using part (2) of Lemma 4.9 and (4.7) shows $\tilde{S}_{2n}(h + \pi i \tau) = (-1)^n \frac{\tilde{S}_{2n}(h)}{\Lambda^{(2n)^2}}$, and thus

$$(6.14) \quad S_{2n}\left(\frac{1}{\lambda}, \frac{x}{\lambda^2}\right) = -S_{2n}\left(\frac{1}{\lambda}, -\frac{x}{\lambda^2}\right) = (-1)^{n-1} \frac{1}{\lambda^{(2n)^2}} S_{2n}(\lambda, x).$$

Similarly using parts (3) and (4) of Lemma 4.9 we get by the same arguments

$$\tilde{R}_{2n+1}(h + \pi i \tau) = \frac{\tilde{\theta}_4((2n+1)h + \pi i(2n+1)\tau)}{\tilde{\theta}_4(h + \pi i \tau)^{(2n+1)^2}} = (-1)^n \frac{\tilde{\theta}_1((2n+1)h)}{\tilde{\theta}_1(h)^{(2n+1)^2}} = (-1)^n \frac{\tilde{S}_{2n+1}}{\Lambda^{(2n+1)^2}},$$

and thus

$$R_{2n+1}\left(\frac{1}{\lambda}, \frac{x}{\lambda^2}\right) = R_{2n+1}\left(\frac{1}{\lambda}, -\frac{x}{\lambda^2}\right) = \frac{(-1)^n}{\lambda^{(2n+1)^2}} S_{2n+1}.$$

The same argument shows $S_{2n+1}\left(\frac{1}{\lambda}, \frac{x}{\lambda^2}\right) = S_{2n+1}\left(\frac{1}{\lambda}, -\frac{x}{\lambda^2}\right) = \frac{(-1)^n}{\lambda^{(2n+1)^2}} R_{2n+1}$. \square

6.3. Blowdown formulas. Let \hat{X} be the blowup of a rational surface X in a point. As mentioned at the beginning of this section, the blowup polynomials determine a blowup formula which computes the K -theoretic Donaldson invariants \hat{X} in terms of those of X . We will also need a blowdown formula which determines all the K -theoretic Donaldson invariants of X in terms of a small part of those of \hat{X} . In order to prove the blowdown formula, we will need that, for n, m relatively prime integers, the polynomials R_n, R_m and S_n, S_m are as polynomials in x in a suitable sense relatively prime.

Proposition 6.15. *Let $n, m \in \mathbb{Z}$ be relatively prime.*

- (1) *There exists a minimal integer $M_{n,m}^0 \in \mathbb{Z}_{\geq 0}$ and unique polynomials $h_{n,m}^0, l_{n,m}^0 \in \mathbb{Q}[\lambda^4, x^2]$, such that $(1 - \lambda^4)^{M_{n,m}^0} = h_{n,m}^0 R_n + l_{n,m}^0 R_m$.*
- (2) *There exists a minimal integers $M_{n,m}^1 \in \mathbb{Z}_{\geq 0}$ and unique polynomials $h_{n,m}^1, l_{n,m}^1 \in \mathbb{Q}[\lambda^4, x]$, such that $\lambda(1 - \lambda^4)^{M_{n,m}^1} = h_{n,m}^1 S_n + l_{n,m}^1 S_m$.*

Proof. For all $l \in \mathbb{Z}$ we write

$$\bar{S}_{2l} := x \frac{S_{2l}}{\lambda} = \frac{S_2 S_{2l}}{S_1^2}, \quad \bar{S}_{2l+1} := \frac{S_{2l+1}}{\lambda} = \frac{S_{2l+1}}{S_1} \in \mathbb{Z}[\lambda^4, x^2].$$

Let $I_{n,m} = \langle R_n, R_m \rangle \subset \mathbb{Z}[\lambda^4, x^2]$ be the ideal generated by $R_n, R_m \in \mathbb{Z}[\lambda^4, x^2]$, and let $J_{n,m} = \langle \bar{S}_n, \bar{S}_m \rangle \subset \mathbb{Z}[\lambda^4, x^2]$ be the ideal generated by $\bar{S}_n, \bar{S}_m \in \mathbb{Z}[\lambda^4, x^2]$. Then the Proposition follows immediately from the following.

Claim (1). There are $M_{n,m}^0, M_{n,m}^1 \in \mathbb{Z}_{\geq 0}$ with $(1 - \lambda^4)^{M_{n,m}^0} \in I_{n,m}$ and $(1 - \lambda^4)^{M_{n,m}^1} \in J_{n,m}$.

Let

$$V_{n,m} := \{(\alpha^4, \beta^2) \in \mathbb{C}^2 \mid R_n(\alpha, \beta) = R_m(\alpha, \beta) = 0\},$$

$$W_{n,m} := \{(\alpha^4, \beta^2) \in \mathbb{C}^2 \mid \bar{S}_n(\alpha, \beta) = \bar{S}_m(\alpha, \beta) = 0\}.$$

Then by the Nullstellensatz the Claim (1) follows immediately from the following.

Claim (2). $V_{n,m}, W_{n,m} \subset \{(1, 0)\}$,

Proof of Claim(2): The idea of the proof is as follows: For each $(\alpha, \beta) \in \mathbb{C}^2$ with $(\alpha^4, \beta^2) \in \mathbb{C}^2 \setminus \{(1, 0)\}$ we want to show that

- (1) $R_n(\alpha, \beta)$ or $R_m(\alpha, \beta)$ is nonzero,
- (2) $\bar{S}_n(\alpha, \beta)$ or $\bar{S}_m(\alpha, \beta)$ is nonzero.

Recall that we have $\tilde{R}_n = R_n(\Lambda, M)$, and we put $\hat{S}_n := \bar{S}_n(\Lambda, M)$, so that $\hat{S}_{2n} = \frac{M\bar{S}_{2n}}{\Lambda}$ and $\hat{S}_{2n+1} = \frac{\bar{S}_{2n+1}}{\Lambda}$. We denote

$$\begin{aligned} \Lambda|_S(h, \tau) &:= \Lambda\left(\frac{h}{\tau}, -\frac{1}{\tau}\right), & M|_S(h, \tau) &:= M\left(\frac{h}{\tau}, -\frac{1}{\tau}\right), \\ \tilde{R}_m|_S(h, \tau) &:= \tilde{R}_m\left(\frac{h}{\tau}, -\frac{1}{\tau}\right), & \hat{S}_m|_S(h, \tau) &:= \hat{S}_m\left(\frac{h}{\tau}, -\frac{1}{\tau}\right) \end{aligned}$$

the application of the operator $S : (h, \tau) \mapsto (\frac{h}{\tau}, -\frac{1}{\tau})$ to the Jacobi functions $\Lambda, M, \tilde{R}_m, \hat{S}_m$. Obviously we have

$$R_m(\Lambda|_S, M|_S) = \tilde{R}_m|_S, \quad \bar{S}_m(\Lambda|_S, M|_S) = \hat{S}_m|_S.$$

We denote $Z(f) \subset \mathbb{C}$ the zero set of a meromorphic function $f : \mathbb{C} \rightarrow \mathbb{C}$. Therefore Claim (2) will follow once we prove the following facts:

- (1) Every $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(1, 0)\}$ can be written as $(\Lambda(h, \tau)^4, M(h, \tau)^2)$ for some $h \in \mathbb{C}$, $\tau \in \mathcal{H} \cup \{\infty\}$ or as $(\Lambda^4|_S(h, \infty), M^2|_S(h, \infty))$ for some $h \in \mathbb{C}$. Here we by $\Lambda(h, \infty)$, $M(h, \infty)$, $(\Lambda|_S(h, \infty), M|_S(h, \infty))$, we mean the coefficient of q^0 of the q -development of $\Lambda, M, (\Lambda|_S, M|_S)$ (asserting also that these developments are power series in q).
- (2) For all $\tau \in \mathcal{H} \cup \{\infty\}$ we have

$$\begin{aligned} Z(\tilde{R}_n(\bullet, \tau)) \cap Z(\tilde{R}_m(\bullet, \tau)) &:= \{h \in \mathbb{C} \mid \tilde{R}_n(h, \tau) = \tilde{R}_m(h, \tau) = 0\} = \emptyset, \\ Z(\tilde{R}_n|_S(\bullet, \infty)) \cap Z(\tilde{R}_m|_S(\bullet, \infty)) &:= \{h \in \mathbb{C} \mid \tilde{R}_n|_S(h, \infty) = \tilde{R}_m|_S(h, \infty) = 0\} = \emptyset. \end{aligned}$$

- (3) For all $\tau \in \mathcal{H} \cup \{\infty\}$ we have

$$\begin{aligned} Z(\hat{S}_n(\bullet, \tau)) \cap Z(\hat{S}_m(\bullet, \tau)) &:= \{h \in \mathbb{C} \mid \hat{S}_n(h, \tau) = \hat{S}_m(h, \tau) = 0\} = \emptyset, \\ Z(\hat{S}_n|_S(\bullet, \infty)) \cap Z(\hat{S}_m|_S(\bullet, \infty)) &:= \{h \in \mathbb{C} \mid \hat{S}_n|_S(h, \infty) = \hat{S}_m|_S(h, \infty) = 0\} = \emptyset. \end{aligned}$$

(1) For any fixed $\tau \in \mathcal{H}$ the range of the elliptic function $\Lambda = \Lambda(\tau, \bullet)$ is $\mathbb{C} \cup \infty$. u is a Hauptmodul for $\Gamma^0(4)$, which takes the values $-2, 2, \infty$ at the cusps $0, 2, \infty$ respectively. Therefore the range of u as a function on \mathcal{H} is $\mathbb{C} \setminus \{-2, 2\}$. By the equation $M^2 = 4(1 + u\Lambda^2 + \Lambda^4)$, we get therefore that the range of (Λ^4, M^2) on $\mathcal{H} \times \mathbb{C}$ contains the set

$$I_1 := \mathbb{C}^2 \setminus \{(c^2, 4(1 + c)^2) \mid c \in \mathbb{C}\}.$$

Now we look at $\tau = \infty$, i.e. $q = 0$. By the q -developments (4.1), we see that

$$(6.16) \quad \begin{aligned} \tilde{\theta}_1(h, \tau) &= O(q), \quad \tilde{\theta}_4(h, \tau) = 1 + O(q), \quad \tilde{\theta}_3(h, \tau) = 1 + O(q), \\ \tilde{\theta}_2(h, \tau) &= \cosh(h/2) + O(q), \quad \frac{\tilde{\theta}_1(h, \tau)}{\tilde{\theta}_2(h, \tau)} = -i \tanh(h/2) + O(q). \end{aligned}$$

Therefore we get from the definitions

$$\Lambda(h, \tau) = O(q), \quad M(h, \tau) = 2 \frac{\tilde{\theta}_2(h, \tau) \tilde{\theta}_3(h, \tau)}{\tilde{\theta}_4(h, \tau)^2} = 2 \cosh(h/2) + O(q).$$

As $\cosh : \mathbb{C} \rightarrow \mathbb{C}$ is surjective, we see that the range of (Λ^4, M^2) on $\mathbb{C} \times \{\infty\}$ is $I_2 := \{0\} \times \mathbb{C}$. From the definitions and (6.16) we obtain

$$\begin{aligned} \Lambda^4|_S(h, \tau) &= \frac{\tilde{\theta}_1(h, \tau)^4}{\tilde{\theta}_2(h, \tau)^4} = \tanh(h/2)^4 + O(q), \\ M^2|_S(h, \tau) &= 4 \frac{\tilde{\theta}_3(h, \tau)^2 \tilde{\theta}_4(h, \tau)^2}{\tilde{\theta}_2(h, \tau)^4} = \frac{4}{\cosh(h/2)^4} + O(q) = 4(1 - \tanh(h/2)^2)^2 + O(q). \end{aligned}$$

It is an easy exercise that the range of $\tanh : \mathbb{C} \rightarrow \mathbb{C}$ is $\mathbb{C} \setminus \{\pm 1\}$. Thus the range of $(\Lambda^4|_S, M^2|_S)$ on $\mathbb{C} \times \{\infty\}$ is

$$I_3 = \{(c^2, 4(1 - c)^2) \mid c \in \mathbb{C} \setminus \{1\}\}.$$

As $I_1 \cup I_2 \cup I_3 = \mathbb{C}^2 \setminus \{(1, 0)\}$, (1) follows.

(2) First let τ in \mathcal{H} . It is standard that $\theta_1(h)$ and $\theta_4(h)$ are holomorphic in h on \mathbb{C} and $Z(\theta_1(h)) = 2\pi i(\mathbb{Z} + \mathbb{Z}\tau)$, $Z(\theta_4(h)) = 2\pi i(\mathbb{Z} + (\mathbb{Z} + \frac{1}{2})\tau)$. Thus by $\tilde{R}_n = \frac{\tilde{\theta}_4(nz)}{\tilde{\theta}_4(z)^{n^2}}$, we see that

$$Z(\tilde{R}_n(\bullet, \tau)) = \left\{ 2\pi i \left(\frac{a}{n} + \frac{b}{2n} \tau \right) \mid a, b \in \mathbb{Z}, b \text{ odd}, (a, b) \not\equiv (0, 0) \pmod{n} \right\}$$

Assume that $2\pi i \left(\frac{a}{n} + \frac{b}{2n} \tau \right) = 2\pi i \left(\frac{a'}{m} + \frac{b'}{2m} \tau \right) \in \mathbb{Z}(\tilde{R}_n(\bullet, \tau)) \cap \mathbb{Z}(\tilde{R}_m(\bullet, \tau))$. As n and m are relatively prime, we see that there $a'', b'' \in \mathbb{Z}$, such that

$$\frac{b}{2n} = \frac{b'}{2m} = \frac{b''}{2}, \quad \frac{a}{n} = \frac{a'}{m} = a''.$$

Thus a and b are both divisible by n , and thus $2\pi i \left(\frac{a}{n} + \frac{b}{2n} \tau \right) \notin \mathbb{Z}(\tilde{R}_n(\bullet, \tau))$.

Now let $\tau = \infty$. Then $\tilde{R}_n(h, \infty) = \frac{\tilde{\theta}_4(nh, \infty)}{\tilde{\theta}_4(h, \infty)^{n^2}} = 1$. Thus $Z(R_n(\bullet, \infty)) = \emptyset$.

Finally we consider $\tilde{R}_n|_S(h, \infty)$. We have

$$\tilde{R}_n|_S(h, \infty) = \frac{\tilde{\theta}_2(nh, \infty)}{\tilde{\theta}_2(h, \infty)^{n^2}} = \frac{\cosh(nh/2)}{\cosh(h/2)^{n^2}},$$

This gives

$$Z(\tilde{R}_n|_S(\bullet, \infty)) = \left\{ \pi i \frac{b}{2n} \mid b \in \mathbb{Z} \text{ odd}, n \nmid b \right\},$$

and again it is clear that $Z(\tilde{R}_n|_S(\bullet, \infty)) \cap Z(\tilde{R}_m|_S(\bullet, \infty)) = \emptyset$.

(3) We note that

$$\begin{aligned}\widehat{S}_{2l+1} &= \frac{\widetilde{S}_{2l+1}}{\widetilde{S}_1} = \frac{\theta_1((2l+1)h)}{\theta_1(h)\widetilde{\theta}_4(h)^{4l^2+4l}}, \\ \widehat{S}_{2l} &= \frac{\widetilde{S}_2\widetilde{S}_{2l}}{\widetilde{S}_1^2} = \frac{\theta_1(2lh)\theta_1(2h)}{\theta_1(h)^2\widetilde{\theta}_4(h)^{4l^2+2}}.\end{aligned}$$

Let $\tau \in \mathcal{H}$, then this gives

$$\begin{aligned}Z(\widehat{S}_{2l+1}(\bullet, \tau)) &= \left\{ 2\pi i \left(\frac{a}{2l+1} + \frac{b}{2l+1} \tau \right) \mid a, b \in \mathbb{Z}, (a, b) \not\equiv (0, 0) \pmod{2l+1} \right\}, \\ Z(\widehat{S}_{2l}(\bullet, \tau)) &= \left\{ 2\pi i \left(\frac{a}{2l} + \frac{b}{2l} \tau \right) \mid a, b \in \mathbb{Z}, (a, b) \not\equiv (0, 0) \pmod{2l} \right\}.\end{aligned}$$

Thus we see immediately that $Z(\widehat{S}_n(\bullet, \tau)) \cap Z(\widehat{S}_m(\bullet, \tau)) = \emptyset$, if n and m are relatively prime.

Now let $\tau = \infty$. Then

$$\widehat{S}_{2l+1}(h, \infty) = \frac{\sinh((2l+1)h/2)}{\sinh(h/2)}, \quad \widehat{S}_{2l}(h, \infty) = \frac{\sinh(lh) \sinh(h)}{\sinh(h/2)^2},$$

So it is easy to see that $Z(\widehat{S}_n(\bullet, \infty)) \cap Z(\widehat{S}_m(\bullet, \infty)) = \emptyset$, if n and m are relatively prime.

Finally

$$\widehat{S}_{2l+1}|_S(h, \tau) = \frac{\theta_1((2l+1)h)}{\theta_1(h)\widetilde{\theta}_2(h)^{4l^2+4l}}, \quad \widehat{S}_{2l}|_S(h, \tau) = \frac{\theta_1(2lh)\theta_1(2h)}{\theta_1(h)^2\widetilde{\theta}_2(h)^{4m^2+2}}.$$

Thus we get

$$\widehat{S}_{2l+1}|_S(h, \infty) = \frac{\sinh((2l+1)h/2)}{\sinh(h/2) \cosh(h/2)^{4l^2+4l}}, \quad \widehat{S}_{2l}|_S(h, \infty) = \frac{\sinh(lh) \sinh(h)}{\sinh(h/2)^2 \cosh(h/2)^{4l^2+2}},$$

and again it is evident that for n and m relatively prime $Z(\widehat{S}_n|_S(\bullet, \infty)) \cap Z(\widehat{S}_m|_S(\bullet, \infty)) = \emptyset$. \square

Corollary 6.17. *Let $n, m \in \mathbb{Z}$ be relatively prime. Let X be \mathbb{P}^2 , $\mathbb{P}^1 \times \mathbb{P}^1$ or the blowup of \mathbb{P}^2 in finitely many general points. Let $c_1 \in H^2(X, \mathbb{Z})$, let L be a line bundle on X , let $r \in \mathbb{Z}_{\geq 0}$ with $\langle c_1, L \rangle \equiv r \pmod{2}$. Let \widehat{X} be the blowup of X in a point. Let $\omega \in S_X$. Using the notations of Proposition 6.15, we have*

$$\begin{aligned}\chi_{c_1}^{X, \omega}(L, P^r) &= \frac{1}{(1 - \Lambda^4)^{M_{n,m}^0}} (\chi_{c_1}^{\widehat{X}, \omega}(L - (n-1)E, P^r \cdot h_{m,n}^0(\Lambda, P)) \\ &\quad + \chi_{c_1}^{\widehat{X}, \omega}(L - (m-1)E, P^r \cdot l_{m,n}^0(\Lambda, P))) \\ &= \frac{1}{\Lambda(1 - \Lambda^4)^{M_{n,m}^1}} (\chi_{c_1+E}^{\widehat{X}, \omega}(L - (n-1)E, P^r \cdot h_{n,m}^1(\Lambda, P)) \\ &\quad + \chi_{c_1+E}^{\widehat{X}, \omega}(L - (m-1)E, P^r \cdot l_{n,m}^1(\Lambda, P))).\end{aligned}$$

Proof. (1) By Theorem 6.7 we have

$$\begin{aligned}&(\chi_{c_1}^{\widehat{X}, \omega}(L - (n-1)E, P^r \cdot h_{m,n}^0(\Lambda, P)) + \chi_{c_1}^{\widehat{X}, \omega}(L - (m-1)E, P^r \cdot l_{m,n}^0(\Lambda, P))) \\ &= \chi_{c_1}^{X, \omega}(L, P^r \cdot (R_n(\Lambda, P)h_{m,n}^0(\Lambda, P) + R_m(\Lambda, P)l_{n,m}^0(\Lambda, P))) = (1 - \Lambda^4)^{M_{n,m}^0} \chi_{c_1}^{X, \omega}(L, P^r),\end{aligned}$$

where in the last step we use Proposition 6.15. The proof of (2) is similar. \square

7. RECURSION FORMULAS FOR RATIONAL RULED SURFACES

7.1. The limit of the invariant at the boundary point. For $X = \mathbb{P}^1 \times \mathbb{P}^1$ or $X = \widehat{\mathbb{P}}^2$ the blowup of \mathbb{P}^2 in a point, we denote the line bundles on X in a uniform way.

Notation 7.1. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ or $X = \widehat{\mathbb{P}}^2$. In the case $X = \mathbb{P}^1 \times \mathbb{P}^1$ we denote F the class of the fibre of the projection to the first factor, and by G the class of the fibre of the projection to the second factor. In the case $X = \widehat{\mathbb{P}}^2$, let H be the pullback of the hyperplane class on \mathbb{P}^2 and E the class of the exceptional divisor. Then $F := H - E$ is the fibre of the ruling of X . We put $G := \frac{1}{2}(H + E)$. Note that G is not an integral cohomology class. In fact, while $H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z}) = \mathbb{Z}F \oplus \mathbb{Z}G$, we have

$$H^2(\widehat{\mathbb{P}}^2, \mathbb{Z}) = \mathbb{Z}H \oplus \mathbb{Z}E = \left\{ aF + bG \mid a \in \mathbb{Z}, b \in 2\mathbb{Z} \text{ or } a \in \mathbb{Z} + \frac{1}{2}, b \in 2\mathbb{Z} + 1 \right\}.$$

On the other hand we note that both on $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $\widehat{\mathbb{P}}^2$ we have $F^2 = G^2 = 0$, $\langle F, G \rangle = 1$, and $-K_X = 2F + 2G$.

We want to define and study the limit of the K -theoretic Donaldson invariants $\chi_{c_1}^{X, \omega}(L, P^r)$ as the ample class ω tends to F . For $c_1 = F$ or $c_1 = 0$ this will be different from our previous definition of $\chi_{c_1}^{X, F}(L, P^r)$.

Definition 7.2. Let $r \in \mathbb{Z}_{\geq 0}$, let $L \in \text{Pic}(X)$ with $\langle c_1, L \rangle + r$ even. Fix $d \in \mathbb{Z}$ with $d \equiv -c_1^2 \pmod{4}$. For $n_{d,r} > 0$ sufficiently large, $n_{d,r}F + G$ is ample on X , and there is no wall ξ of type (c_1, d) with $\langle \xi, (n_{d,r}dF + G) \rangle > 0 > \langle \xi, F \rangle$. For all $\omega \in S_X \cup C_X$, we define $\chi_{c_1, d}^{X, \omega}(L, P^r) := \text{Coeff}_{\Lambda^d} [\chi_{c_1, d}^{X, \omega}(L, P^r)]$, and put

$$\chi_{c_1, d}^{X, F_+}(L, P^r) := \chi_{c_1, d}^{X, n_{d,r}F + G}(L, P^r), \quad \chi_{c_1}^{X, F_+}(L) := \sum_{d \geq 0} \chi_{c_1, d}^{X, F_+}(L, P^r) \Lambda^d.$$

Now we give a formula for $\chi_0^{X, F_+}(nF + mG, P^r)$ and $\chi_F^{X, F_+}(nF + mG, P^r)$. The result and the proof are similar to [10, Prop. 5.3]. The rest of this section will be mostly devoted to giving an explicit evaluation of this formula for $m \leq 2$.

Proposition 7.3. *Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ or $X = \widehat{\mathbb{P}}^2$.*

(1) *Let $nF + mG$ be a line bundle on X with m even. Then*

$$\chi_F^{X, F_+}(nF + mG, P^r) = \text{Coeff}_{q^0} \left[\frac{1}{2 \sinh((m/2 + 1)h)} \Lambda^2 \widetilde{\theta}_4(h)^{2(n+2)(m+2)} u' h^* M^r \right].$$

(2) *Let $nF + mG$ be a line bundle on X (note that we might have $n \in \frac{1}{2}\mathbb{Z}$). Then*

$$\chi_0^{X, F_+}(nF + mG, P^r) = - \text{Coeff}_{q^0} \left[\frac{1}{2} (\coth((m/2 + 1)h)) \Lambda^2 \widetilde{\theta}_4(h)^{2(n+2)(m+2)} u' h^* M^r \right].$$

Proof. We denote $\Gamma_X = H^2(X, \mathbb{Z})$ with inner product the negative of the intersection form. Let $c_1 = 0$ or $c_1 = F$, fix d , and let $s \in \mathbb{Z}_{\geq 0}$ be sufficiently large so that there is no class ξ of (c_1, d) with $\langle \xi, F \rangle < 0 < \langle \xi, (G + sF) \rangle$. Write $L := nF + mG$. By Corollary 6.6 we get

$$\begin{aligned} \chi_{F,d}^{X,sF+G}(L, P^r) &= \text{Coeff}_{\Lambda^d} \text{Coeff}_{q^0} \left[\Psi_{X,F}^{G+sF,F}(L; \Lambda, \tau) M^r \right] \\ &= \text{Coeff}_{\Lambda^d} \text{Coeff}_{q^0} \left[\Theta_{\Gamma_X, F, K_X}^{G+sF,F} \left(\frac{1}{2\pi i} (L - K_X)h, \tau \right) \Lambda^2 \tilde{\theta}_4(h)^{(L-K_X)^2} u' h^* M^r \right] \\ &= \text{Coeff}_{\Lambda^d} \text{Coeff}_{q^0} \left[\frac{e^{-\langle \frac{F}{2}, (L-K_X) \rangle}}{1 - e^{-\langle F, (L-K_X) \rangle h}} \Lambda^2 \tilde{\theta}_4(h)^{(L-K_X)^2} u' h^* M^r \right] + \sum_{\langle F, \xi \rangle < 0 < \langle (G+sF), \xi \rangle} \delta_\xi^X(L, P^r). \end{aligned}$$

Here the second sum is over the classes of type (F, d) . By our assumption on s the second sum is empty, so we get

$$\begin{aligned} \chi_F^{X,F+}(L, P^r) &= \text{Coeff}_{q^0} \left[\frac{e^{-\langle \frac{F}{2}, (L-K_X) \rangle h}}{1 - e^{-\langle F, (L-K_X) \rangle h}} \Lambda^2 \tilde{\theta}_4(h)^{(L-K_X)^2} u' h^* M^r \right] \\ &= \text{Coeff}_{q^0} \left[\frac{\Lambda^2 \tilde{\theta}_4(h)^{(L-K_X)^2} u' h^*}{2 \sinh(\langle \frac{F}{2}, (L-K_X) \rangle h)} M^r \right]. \end{aligned}$$

In the case $c_1 = 0$ the argument is very similar. By definition and Theorem 5.1 we have

$$\begin{aligned} \chi_{0,d}^{X,sF+G}(L, P^r) &= \text{Coeff}_{\Lambda^d} \text{Coeff}_{q^0} \left[\Psi_{X,0}^{sF+G,F}(L; \Lambda, \tau) M^r \right] \\ &= \text{Coeff}_{\Lambda^d} \text{Coeff}_{q^0} \left[\Theta_{\Gamma_X, 0, K_X}^{sF+G,F} \left(\frac{1}{2\pi i} (L - K_X)h, \tau \right) \Lambda^2 \tilde{\theta}_4(h)^{(L-K_X)^2} u' h^* M^r \right] \\ &= - \text{Coeff}_{\Lambda^d} \text{Coeff}_{q^0} \left[\frac{e^{-\langle F, L-K_X \rangle h} \Lambda^2 \tilde{\theta}_4(h)^{(L-K_X)^2} u' h^*}{1 - e^{-\langle F, (L-K_X) \rangle h}} M^r \right] + \sum_{\langle F, \xi \rangle < 0 < \langle (G+sF), \xi \rangle} \delta_\xi^X(L, P^r). \end{aligned}$$

The second sum is again over the walls of type $(0, d)$, and thus it is 0. Thus we get

$$\begin{aligned} \chi_0^{X,F+}(L, P^r) &= - \text{Coeff}_{q^0} \left[\frac{e^{-\langle F, L-K_X \rangle h} \Lambda^2 \tilde{\theta}_4(h)^{(L-K_X)^2} u' h^*}{1 - e^{-\langle F, (L-K_X) \rangle h}} M^r \right] \\ &= - \text{Coeff}_{q^0} \left[\frac{1}{2} (\coth(\langle F, (L-K_X)/2 \rangle h) - 1) \Lambda^2 \tilde{\theta}_4(h)^{(L-K_X)^2} u' h^* M^r \right]. \end{aligned}$$

Note that by Remark 4.11, we get

$$\text{Coeff}_{q^0} [\Lambda^2 \tilde{\theta}_4(h)^{(L-K_X)^2} u' h^* M^r] = \text{Coeff}_{q^0} [(1-1) \Lambda^2 \tilde{\theta}_4(h)^{(L-K_X)^2} u' h^* M^r] = 0.$$

□

Remark 7.4. In the case of $\mathbb{P}^1 \times \mathbb{P}^1$, we can in the same way define $\chi_{c_1,d}^{\mathbb{P}^1 \times \mathbb{P}^1, G_+}(L, P^r) := \chi_{c_1,d}^{\mathbb{P}^1 \times \mathbb{P}^1, G_+ + n_d F}(L, P^r)$ for n_d sufficiently large with respect to d , and

$$\chi_{c_1}^{\mathbb{P}^1 \times \mathbb{P}^1, G_+}(nF + mG, P^r) := \sum_{d > 0} \chi_{c_1,d}^{\mathbb{P}^1 \times \mathbb{P}^1, G_+}(L, P^r) \Lambda^d.$$

Then we see immediately that $\chi_F^{\mathbb{P}^1 \times \mathbb{P}^1, G+}(nF + mG, P^r) = 0$, and we get by symmetry from Proposition 7.3 that

$$\chi_0^{\mathbb{P}^1 \times \mathbb{P}^1, G+}(nF + mG, P^r) = -\text{Coeff}_{q^0} \left[\frac{1}{2} (\coth((n/2 + 1)h)) \Lambda^2 \tilde{\theta}_4(h)^{2(n+2)(m+2)} u' h^* M^r \right].$$

7.2. Recursion formulas from theta constant identities. We now use the blowup polynomials to show recursion formulas in n and r for the K -theoretical Donaldson invariants $\chi_0^{X, F+}(nF + mG, P^r)$, $\chi_F^{X, F+}(nF + mG, P^r)$ for $0 \leq m \leq 2$. We use the fact that the \tilde{S}_n vanish at division points $a \in \frac{2}{n}\pi i\mathbb{Z}$, together with other vanishing results proven in [10]. We consider expressions relating the left hand sides of the formulas of Proposition 7.3 for $\chi_0^{X, F+}(nF + mG, P^r)$, $\chi_F^{X, F+}(nF + mG, P^r)$ for successive values of n . We will show that these are almost holomorphic in q , i.e. that they have only finitely many monomials $\Lambda^d q^s$ with nonzero coefficients and $s \leq 0$. This will then give recursion formulas for $\chi_0^{X, F+}(nF + mG, P^r)$, $\chi_F^{X, F+}(nF + mG, P^r)$.

We will frequently use the following

- Notation 7.5.** (1) For a power series $f = \sum_{n \geq 0} f_n(y)q^n \in \mathbb{C}[y^{\pm 1}][[q]]$, and a polynomial $g \in \mathbb{C}[y^{\pm 1}]$ we say that g divides f , if g divides f_n for all n .
- (2) For a Laurent series $h = \sum_n a_n q^n \in \mathbb{C}((q))$ the *principal part* is $\mathcal{P}[h] := \sum_{n \leq 0} a_n q^n$. Note that this contains the coefficient of q^0 . This is because we think of q as $e^{\pi i \tau / 4}$, with τ in \mathcal{H} , and then $\frac{dq}{q} = \frac{\pi i}{4} d\tau$. For a series $h = \sum_{n \geq 0} h_n(q) \Lambda^n \in \mathbb{C}((q))[[\Lambda]]$, the *principal part* is $\mathcal{P}[h] := \sum_{n \geq 0} \mathcal{P}[h_n] \Lambda^n \in \mathbb{C}((q))[[\Lambda]]$. We recall the previous notation $\text{Coeff}_{q^0}[h] := \sum_{n \geq 0} \text{Coeff}_{q^0}[h_n] \Lambda^n$.
- (3) We write $\mathbb{Q}[[y^{\pm 2} q^4, q^4]]^\times$ for the set power series in $y^{\pm 2} q^4, q^4$ whose constant part is 1.

Remark 7.6. By (4.2), (4.3) we have $\mathcal{P}[M^2] = 4 - q^{-2} \Lambda^2 + 4 \Lambda^4$, and thus obviously

$$\begin{aligned} \mathcal{P}[M^2 - (1 - \Lambda^4)^2] &= 3 - q^{-2} \Lambda^2 + 6 \Lambda^4 - \Lambda^8, \\ \mathcal{P}[M^2(1 + \Lambda^4) - 2(1 - \Lambda^4)^2] &= 2 - q^{-2} \Lambda^2 + 12 \Lambda^4 - q^{-2} \Lambda^6 + 2 \Lambda^8. \end{aligned}$$

Lemma 7.7. *For all $r \in \mathbb{Z}_{>0}$ we have*

- (1) $g_1^r := \mathcal{P} \left[\frac{1}{2 \sinh(h)} M^{2r} u' h^* \Lambda^2 \right] \in \mathbb{Q}[q^{-2} \Lambda^2, \Lambda^4]_{\leq r}$,
- (2) $g_2^r := \mathcal{P} \left[-\frac{1}{2} \coth(h) M^{2r} u' h^* \Lambda^2 \right] \in \mathbb{Q}[q^{-2} \Lambda^2, \Lambda^4]_{\leq r+1}$,
- (3) $\mathcal{P} \left[\frac{1}{2 \sinh(3h/2)} M(\tilde{\theta}_4(h)^3(1 - \Lambda^4) - 1) u' h^* \Lambda^2 \right] = \Lambda^4$,
- (4) $g_3^r := \mathcal{P} \left[\frac{1}{2 \sinh(3h/2)} M^{2r-1} (M^2 \tilde{\theta}_4(h)^3 - (1 - \Lambda^4)) u' h^* \Lambda^2 \right] \in \mathbb{Q}[q^{-2} \Lambda^2, \Lambda^4]_{\leq r}$,
- (5) $g_4^r := \mathcal{P} \left[-\frac{1}{2} \coth(3h/2) M^{2r-2} (M^2 \tilde{\theta}_4(h)^3 - (1 - \Lambda^4)) u' h^* \Lambda^2 \right] \in \mathbb{Q}[q^{-2} \Lambda^2, \Lambda^4]_{\leq r+1}$,
- (6) $g_5^r := \mathcal{P} \left[-\frac{1}{2} \tanh(h) M^{2r-2} (\tilde{\theta}_4(h)^8 (M^2 - (1 - \Lambda^4)^2) - 1) u' h^* \Lambda^2 \right] \in \mathbb{Q}[q^{-2} \Lambda^2, \Lambda^4]_{\leq r+2}$.

Proof. (1) We know $\tilde{\theta}_4(h) \in \mathbb{Q}[[y^{\pm 2}q^4, q^4]]^\times$, $\tilde{\theta}_1(h) \in iq(y - y^{-1})\mathbb{Q}[[y^{\pm 2}q^4, q^4]]^\times$, and from the product formula of (4.1) we see that even $\tilde{\theta}_1(2h) \in iq(y^2 - y^{-2})\mathbb{Q}[[y^{\pm 2}q^4, q^4]]^\times$. By Definition 6.1, Proposition 6.3 we have $\frac{\tilde{\theta}_1(2h)}{\tilde{\theta}_4(h)^4} = \Lambda M$, thus we get that $\Lambda^2 M^2 \in q^2(y^2 - y^{-2})^2 \mathbb{Q}[[y^{\pm 2}q^4, q^4]]$. As $u' \in q^{-2}\mathbb{Q}[[q^4]]$, we get that

$$f(y, q) := \sum_{n \geq 0} f_n(y) q^{4n} := \frac{1}{\sinh(h)} \Lambda^2 M^2 u' \in (y^2 - y^{-2}) \mathbb{Q}[[y^{\pm 2}q^4, q^4]].$$

Thus $f_n(y)$ is a Laurent polynomial in y^2 of degree at most $n + 1$, and we see from the definitions that it is antisymmetric under $y \rightarrow y^{-1}$. Therefore $f_n(y)$ can be written as a linear combination of $\sinh(lh)$ for $l = 1, \dots, n + 1$. Thus we get by Lemma 4.14 that $f_n(y)h^* \in \mathbb{Q}[q^{-2}\Lambda^2]_{\leq n+1} \mathbb{Q}[[q^2\Lambda^2, q^4]]$, and thus the principal part of $q^{4n}f_n(y)h^*$ vanishes unless $4n \leq 2n + 2$, i.e. $n \leq 1$. Therefore the principal part of $f(y, q)h^*$ is a polynomial in $q^{-2}\Lambda^2, \Lambda^2 q^2$, and q^4 and thus (as the power of q must be nonpositive) a polynomial in $q^{-2}\Lambda^2$ and Λ^4 , and we see that its degree is at most 1.

By (4.3), we have that $M^2 = 4 + 4u\Lambda^2 + 4\Lambda^4$. Using that $u \in q^{-2}\mathbb{Q}[[q^4]]$ we get that $M^2 \in \mathbb{Q}[q^{-2}\Lambda^2]_{\leq 1} \mathbb{Q}[[q^2\Lambda^2, q^4]]$. Therefore by the above

$$M^{2r-2}f_n(y, q)h^* \in \mathbb{Q}[q^{-2}\Lambda^2]_{\leq n+r} \mathbb{Q}[[q^2\Lambda^2, q^4]].$$

The same argument as above shows that the principal part of $M^{2r-2}f(y, q)h^*$ is a polynomial in $q^{-2}\Lambda^2$ and Λ^4 of degree at most r .

(2) In (1) we have seen that

$$M^{2r-2}f_n(y, q)h^* \in \mathbb{Q}[q^{-2}\Lambda^2]_{\leq n+r} \mathbb{Q}[[q^2\Lambda^2, q^4]].$$

We have $\coth(h)\Lambda^2 M^{2r}u'h^* = \cosh(h)M^{2r-2}f(y, q)h^*$, and by Lemma 4.14 we have that

$$\cosh(h)M^{2r-2}f_n(y, q)h^* \in \mathbb{Q}[q^{-2}\Lambda^2]_{\leq n+r+1} \mathbb{Q}[[q^2\Lambda^2, q^4]]$$

The same argument as in (1) shows that the principal part of $\cosh(h)M^{2r-2}f(y, q)h^*$ is a polynomial in $q^{-2}\Lambda^2$ and Λ^4 of degree at most $r + 1$.

(3) By [10], Prop. 5.10(5) and its proof, we have that $\tilde{\theta}_4(h)^3(1 - \Lambda^4) - 1 \in \mathbb{Q}[y^{\pm 1}][[q]]$ is divisible by $y^3 - y^{-3}$. Thus also $M(\tilde{\theta}_4(h)^3(1 - \Lambda^4) - 1) \in \mathbb{Q}[y^{\pm 2}][[q]]$ is divisible by $y^3 - y^{-3}$. We note that $\Lambda \in iq(y - y^{-1})\mathbb{Q}[[y^{\pm 2}q^4, q^4]]$, thus $1 - \Lambda^4 \in \mathbb{Q}[y^{\pm 2}]_{\leq 1} \mathbb{Q}[[y^{\pm 2}q^4, q^4]]$. We already know $\tilde{\theta}_4(h) \in \mathbb{Q}[[y^{\pm 2}q^4, q^4]]$, $M \in (y + y^{-1})\mathbb{Q}[[y^{\pm 2}q^4, q^4]]$. Thus $M(\tilde{\theta}_4^3(1 - \Lambda^4) - 1) \in (y^3 - y^{-3})\mathbb{Q}[[y^{\pm 2}q^4, q^4]]$. Therefore, writing

$$f := \sum_{n \geq 0} f_n(y) q^{4n} := \frac{1}{2 \sinh(3h/2)} M(\tilde{\theta}_4(h)^3(1 - \Lambda^4) - 1),$$

$f_n(y)$ is a Laurent polynomial in y^2 of degree at most n , and we see from the definitions that it is antisymmetric under $y \rightarrow y^{-1}$. Thus by Lemma 4.14 we get $f_n(y)h^* \in \mathbb{Q}[q^{-2}\Lambda^2]_{\leq n} \mathbb{Q}[[q^2\Lambda^2, q^4]]$. Therefore $fh^* \in \mathbb{Q}[[q^2\Lambda^2, q^4]]$ and $fh^*u'\Lambda^2 \in [q^{-2}\Lambda^2, \Lambda^4]_{\leq 1} \mathbb{Q}[[q^2\Lambda^2, q^4]]$. Computation of the first few coefficients gives $\mathcal{P}[fu'\Lambda^2] = \Lambda^4$.

(4) As $\tilde{\theta}_4(h)^3(1-\Lambda^4)-1 \in \mathbb{Q}[y^{\pm 1}][[q]]$ is divisible by y^3-y^{-3} , the same is true for $\Lambda M^2(\tilde{\theta}_4^3(1-\Lambda^4)-1)$. On the other hand $\Lambda(M^2-(1-\Lambda^4)^2) \in i\mathbb{Q}[y^{\pm 1}][[q]]$, and by $\Lambda(M^2-(1-\Lambda^4)^2) = \tilde{S}_3 = \frac{\tilde{\theta}_1(3h)}{\theta_4(h)^9}$ (see Definition 6.1, Proposition 6.3), we see that $\Lambda(M^2-(1-\Lambda^4)^2)$, vanishes for $3h \in 2\pi i\mathbb{Z}$, i.e. when $y^3 = y^{-3}$. Thus it is also divisible by y^3-y^{-3} . Therefore also

$$\Lambda(M^2\tilde{\theta}_4(h)^3(1-\Lambda^4)-(1-\Lambda^4)^2) = \Lambda M^2(\tilde{\theta}_4^3(1-\Lambda^4)-1) + \Lambda(M^2-(1-\Lambda^4)^2) \in i\mathbb{Q}[y^{\pm 1}][[q]]$$

is divisible y^3-y^{-3} . We note that $(1-\Lambda^4) = \tilde{R}_2 = \frac{\tilde{\theta}_4(2h)}{\theta_4(h)^4} \in \mathbb{Q}[y^{\pm 1}][[q]]$ does not vanish at any h with $3h \in 2\pi i\mathbb{Z}$. It follows that also the power series $\Lambda(M^2\tilde{\theta}_4(h)^3-(1-\Lambda^4))$ is divisible by y^3-y^{-3} . Finally we note that $\Lambda \in iq(y-y^{-1})\mathbb{Q}[[y^{\pm 2}q^4, q^4]]$, thus $1-\Lambda^4 \in \mathbb{Q}[y^{\pm 2}]_1\mathbb{Q}[[y^{\pm 2}q^4, q^4]]$. Thus

$$\Lambda(M^2\tilde{\theta}_4(h)^3-(1-\Lambda^4)) \in iq(y-y^{-1})\mathbb{Q}[y^{\pm 2}]_{\leq 1}\mathbb{Q}[[y^{\pm 2}q^4, q^4]],$$

and therefore, as it is divisible by y^3-y^{-3} , we can write $\frac{1}{\sinh(3h/2)}M\Lambda(M^2\tilde{\theta}_4(h)^3-(1-\Lambda^4)) = q\sum_{n \geq 0} \bar{f}_n(y)q^{4n}$, where $\bar{f}_n(y)$ is an odd Laurent polynomial in y of degree $2n+1$, symmetric under $y \rightarrow y^{-1}$. Thus by Lemma 4.14 we get $\bar{f}_n(y)h^* \in \Lambda\mathbb{Q}[q^{-2}\Lambda^2]_{\leq n}\mathbb{Q}[[q^2\Lambda^2, q^4]]$, and thus $\bar{f}_n(y)h^*u'\Lambda \in \mathbb{Q}[q^{-2}\Lambda^2]_{\leq n+1}\mathbb{Q}[[q^2\Lambda^2, q^4]]$. It follows as before that the principal part of $\frac{1}{2\sinh(3h/2)}M(M^2\tilde{\theta}_4(h)^3-(1-\Lambda^4))u'h^*\Lambda^2$ is a polynomial in $q^{-2}\Lambda^2$ and Λ^4 of degree at most 1. Using the fact that $M^2 \in \mathbb{Q}[q^{-2}\Lambda^2]_{\leq 1}\mathbb{Q}[[q^2\Lambda^2, q^4]]$ in the same way as in the proof of (1) we see that the principal part of $\frac{1}{2\sinh(3h/2)}M^{2r-1}(M^2\tilde{\theta}_4(h)^3-(1-\Lambda^4))u'\Lambda^2h^*$ is a polynomial of degree at most r in $q^{-2}\Lambda^2$ and Λ^4 .

(5) We see that the left hand side of (5) is obtained from the left hand side of (4) by multiplying by $\cosh(3h/2)/M$. As by the above $M \in (y+y^{-1})\mathbb{Q}[[y^{\pm 2}q^4, q^4]]^\times$, we see that

$$2\cosh(3h/2)/M = (y^3+y^{-3})/M \in (y^2-1+y^{-2})\mathbb{Q}[[y^{\pm 2}q^4, q^4]] \subset \mathbb{Q}[q^{-2}\Lambda^2]_{\leq 1}\mathbb{Q}[[q^2\Lambda^2, q^4]],$$

where the inclusion on the right follows again by Lemma 4.14. Therefore (5) follows from (4).

(6) We note that by Definition 6.1 and Proposition 6.3 we have

$$s_1 := (1+\Lambda^4)M^2-2(1-\Lambda^4)^2 = \frac{S_4(\Lambda, M)}{S_2(\Lambda, M)R_2(\Lambda, M)} = \frac{\tilde{\theta}_1(4h)}{\tilde{\theta}_1(2h)\tilde{\theta}_4(2h)\tilde{\theta}_4(h)^8}.$$

Again s_1 is in $\mathbb{Q}[y^{\pm 1}][[q]]$. As $\tilde{\theta}_4(h)$ has no zeros on $2\pi i\mathbb{R}$ and $\tilde{\theta}_1(h)$ vanishes precisely for $h \in 2\pi i\mathbb{Z}$, we find that s_1 vanishes if $y^4 = y^{-4}$, but not $y^2 = y^{-2}$. Thus the coefficient of every power of q of s_1 is divisible by y^2+y^{-2} .

In [10] Proposition 5.10(6) and its proof it is shown that

$$s_2 := \tilde{\theta}_4(h)^8(1-\Lambda^4)^3-(1+\Lambda^4) \in (y^2+y^{-2})\mathbb{Q}[y^{\pm 1}][[q]].$$

Thus also

$$M^2\tilde{\theta}_4(h)^8(1-\Lambda^4)^3-2(1-\Lambda^4)^2 = M^2s_2+s_1 \in (y^2+y^{-2})\mathbb{Q}[y^{\pm 1}][[q]].$$

As $\tilde{R}_2 = (1 - \Lambda^4) \in \mathbb{Q}[y^{\pm 1}][[q]]^\times$ does not vanish for $h \in i\mathbb{R}$, we get that $s_3 := M^2 \tilde{\theta}_4(h)^8 (1 - \Lambda^4) - 2 \in (y^2 + y^{-2})\mathbb{Q}[y^{\pm 1}][[q]]$. Therefore also

$$\frac{1}{2}(s_3 + \tilde{\theta}_4(h)^8 s_1) = M^2 \tilde{\theta}_4(h)^8 - \tilde{\theta}_4(h)^8 (1 - \Lambda^4)^2 - 1 \in (y^2 + y^{-2})\mathbb{Q}[y^{\pm 1}][[q]].$$

On the other hand we know $M^2 \in (y + y^{-1})^2 \mathbb{Q}[[y^{\pm 2} q^4, q^4]]$, $\tilde{\theta}_4(h) \in \mathbb{Q}[[y^{\pm 2} q^4, q^4]]$ and $(1 - \Lambda^4)^2 \in \mathbb{Q}[y^{\pm 2}]_{\leq 2} \mathbb{Q}[[y^{\pm 2} q^4, q^4]]$. Thus

$$l := \tanh(h) (M^2 \tilde{\theta}_4(h)^8 - \tilde{\theta}_4(h)^8 (1 - \Lambda^4)^2 - 1) \in \mathbb{Q}[y^{\pm 2}]_{\leq 2} \mathbb{Q}[[y^{\pm 2} q^4, q^4]].$$

Thus we can write $l = \sum_{n \geq 0} l_n(y) q^{4n}$ where $l_n(y)$ is a Laurent polynomial in y^2 of degree $n + 2$, symmetric under $y \rightarrow y^{-1}$. Thus by Lemma 4.14 we get $l_n(y) h^* \in \mathbb{Q}[q^{-2} \Lambda^2]_{\leq n+2} \mathbb{Q}[[q^2 \Lambda^2, q^4]]$, and thus $l_n(y) h^* u' \Lambda^2 \in \mathbb{Q}[q^{-2} \Lambda^2]_{\leq n+3} \mathbb{Q}[[q^2 \Lambda^2, q^4]]$. It follows as before that the principal part of $\tanh(h) (M^2 \tilde{\theta}_4(h)^8 - \tilde{\theta}_4(h)^8 (1 - \Lambda^4)^2 - 1) h^* u' \Lambda^2$ is a polynomial in $q^{-2} \Lambda^2$ and Λ^4 of degree at most 3. Using again the fact that $M^2 \in \mathbb{Q}[q^{-2} \Lambda^2]_{\leq 1} \mathbb{Q}[[q^2 \Lambda^2, q^4]]$, we see that the principal part of $\tanh(h) M^{2r-2} (M^2 \tilde{\theta}_4(h)^8 - \tilde{\theta}_4(h)^8 (1 - \Lambda^4)^2 - 1) h^* u' \Lambda^2$ is a polynomial of degree at most $r + 2$ in $q^{-2} \Lambda^2$ and Λ^4 . \square

Remark 7.8. The principal parts above can be easily computed by calculations with the lower order terms with the power series, using the formulas given in §4.1. We see for instance:

$$\begin{aligned} g_1^1 &= q^{-2} \Lambda^2 - 4 \Lambda^4, & g_2^1 &= -q^{-2} \Lambda^2 + \left(\frac{1}{2} q^{-4} - 8\right) \Lambda^4 - q^{-2} \Lambda^6 - \Lambda^8, \\ g_3^1 &= q^{-2} \Lambda^2 - 5 \Lambda^4, & g_3^2 &= 4 q^{-2} \Lambda^2 - (q^{-4} + 20) \Lambda^4 + 9 q^{-2} \Lambda^6 - 23 \Lambda^8, \\ g_4^1 &= -\frac{1}{2} q^{-2} \Lambda^2 + \left(\frac{1}{2} q^{-4} - 11\right) \Lambda^4 - \frac{1}{2} \Lambda^8, \\ g_5^1 &= \left(\frac{1}{2} q^{-4} - 12\right) \Lambda^4 + 2 q^{-2} \Lambda^6 + 4 \Lambda^8 - \frac{1}{2} q^{-2} \Lambda^{10} + 5 \Lambda^{12}. \end{aligned}$$

We apply Lemma 7.7 to compute the limit of the K -theoretic Donaldson invariants at F .

Proposition 7.9. *For $X = \mathbb{P}^1 \times \mathbb{P}^1$ or $X = \hat{\mathbb{P}}^2$, all $n \in \mathbb{Z}$ we have*

- (1) $1 + \chi_F^{X, F+}(nF) = \frac{1}{(1 - \Lambda^4)^{n+1}}$.
- (2) *For all $r > 0$ there is a polynomial $h_r^0(n, \Lambda^4) \in \mathbb{Q}[\Lambda^4, n\Lambda^4]_{\leq r}$ with $\chi_F^{X, F+}(nF, P^{2r}) = h_r^0(n, \Lambda^4)$.*
- (3) $1 + (2n + 5) \Lambda^4 + \chi_0^{X, F+}(nF) = \frac{1}{(1 - \Lambda^4)^{n+1}}$.
- (4) *For all $r > 0$ there is a polynomial $\bar{h}_r^0(n, \Lambda^4) \in \mathbb{Q}[\Lambda^4, n\Lambda^4]_{\leq r+1}$ with $\chi_0^{X, F+}(nF, P^{2r}) = \bar{h}_r^0(n, \Lambda^4)$.*

Proof. (1) and (3) are proven in [10, Prop. 5.14].

(2) Let $r > 0$. By Proposition 7.3 we have

$$\chi_F^{X, F+}(nF, P^{2r}) = \text{Coeff}_{q^0} \left[\frac{1}{2 \sinh(h)} \Lambda^2 \tilde{\theta}_4(h)^{4n+8} u' h^* M^{2r} \right] = \text{Coeff}_{q^0} \left[g_1^r \tilde{\theta}_4(h)^{4n+8} \right].$$

where the last step uses Lemma 7.7(1) and the fact that $\tilde{\theta}_4(h) \in \mathbb{Q}[[q^2\Lambda^2, q^4]]^\times$, and thus $\tilde{\theta}_4(h)^{4n+8} \in \mathbb{Q}[[nq^2\Lambda^2, nq^4, q^2\Lambda^2, q^4]]^\times$. As $g_1^r(q^{-2}\Lambda^2, \Lambda^4)$ is a polynomial of degree at most r , we see that $\text{Coeff}_{q^0} [g_1^r \tilde{\theta}_4(h)^{4n+8}]$ is a polynomial of degree at most r in $\Lambda^4, n\Lambda^4$.

(4) Let $r > 0$. By Proposition 7.3 and Lemma 7.7(2) we have

$$\chi_F^{X, F+}(nF, P^{2r}) = \text{Coeff}_{q^0} \left[-\frac{1}{2} \coth(h) \Lambda^2 \tilde{\theta}_4(h)^{4n+8} u' h^* M^{2r} \right] = \text{Coeff}_{q^0} [g_2^r \tilde{\theta}_4(h)^{4n+8}].$$

As g_2^r is a polynomial of degree at most $r+1$, we see as in (1) that $\text{Coeff}_{q^0} [g_2^r \tilde{\theta}_4(h)^{4n+8}]$ is a polynomial of degree at most $r+1$ in $\Lambda^4, n\Lambda^4$. \square

Remark 7.10. We list the first few polynomials h_r^0, \bar{h}_r^0 .

$$\begin{aligned} h_1^0 &= (4n+4)\Lambda^4, & h_2^0 &= (16n+16)\Lambda^4 - (8n^2+6n-3)\Lambda^8, \\ h_3^0 &= (64n+64)\Lambda^4 + (-64n^2+24n+100)\Lambda^8 + (\frac{32}{3}n^3-8n^2-\frac{68}{3}n)\Lambda^{12}, \\ \bar{h}_1^0 &= -(4n+16)\Lambda^4 + (4n^2+15n+13)\Lambda^8, \\ \bar{h}_2^0 &= -(16n+64)\Lambda^4 + (24n^2+78n+18)\Lambda^8 - (\frac{16}{3}n^3+20n^2+\frac{50}{3}n-2)\Lambda^{12}. \end{aligned}$$

Proposition 7.11. *For $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $n \in \mathbb{Z}$, and for $X = \widehat{\mathbb{P}}^2$ and $n \in \mathbb{Z} + \frac{1}{2}$, and all $r \in \mathbb{Z}_{\geq 0}$ we have the following.*

$$\begin{aligned} (1) \quad \chi_F^{X, F+}(nF+G, P^{2r+1}) &= \frac{1}{(1-\Lambda^4)^{2n+1-2r}} + h_r^1(n, \Lambda^4), \text{ where } h_r^1(n, \Lambda^4) \in \mathbb{Q}[\Lambda^4, n\Lambda^4]_{\leq r}. \\ (2) \quad \chi_0^{X, F+}(nF+G, P^{2r}) &= \frac{1}{(1-\Lambda^4)^{2n+2-2r}} + \bar{h}_r^1(n, \Lambda^4), \text{ where } \bar{h}_r^1(n, \Lambda^4) \in \mathbb{Q}[\Lambda^4, n\Lambda^4]_{\leq r+1}. \end{aligned}$$

Proof. (1) First we deal with the case $r = 0$. We do this by ascending and descending induction on n . Let $n = -1$. By Corollary 6.6 we know that $\chi_F^{X, G}(-F+G, P) = 0$ and

$$\chi_F^{X, F+}(-F+G, P) = \sum_{\xi} \delta_{\xi}^X(-F+G, P),$$

where ξ runs through all classes of type F with $G\xi < 0 < F\xi$, i.e. through all $\xi = (2mG - (2n-1)F)$ with $n, m \in \mathbb{Z}_{>0}$. By Lemma 4.15 we have $\delta_{2mG-(2n-1)F}^X(-F+G, P) = 0$ unless $|6n-3-2m|+3 \geq 8nm-4m$, and we check easily that this can only happen for $n = m = 1$. Then computing with the lowest order terms of the formula of Definition 4.10 gives $\delta_{2G-F}^X(-F+G, P) = -\Lambda^4$. Thus $\chi_F^{X, F+}(-F+G, P) = -\Lambda^4 = (1-\Lambda^4) - 1$. This shows the case $n = -1$.

Now let $n \in \frac{1}{2}\mathbb{Z}$ be general, then we have by Proposition 7.3 that

$$\begin{aligned} (1-\Lambda^4) \chi_F^{X, F+}((n+1/2)F+G, P) &- \chi_F^{X, F+}(nF+G, P) \\ &= \text{Coeff}_{q^0} \left[\frac{1}{2 \sinh(3h/2)} \tilde{\theta}_4(h)^{6(n+2)} (\tilde{\theta}_4(h)^3 (1-\Lambda^4) - 1) M u' h^* \Lambda^2 \right] \\ &= \text{Coeff}_{q^0} [\tilde{\theta}_4(h)^{6(n+2)} \Lambda^4] = \Lambda^4, \end{aligned}$$

and where in the last line we have used Lemma 7.7(3) and the fact that $\tilde{\theta}_4(h) \in \mathbb{Q}[[q^2\Lambda^2, q^4]]^\times$. Thus

$$(1 - \Lambda^4)(1 + \chi_F^{X, F+}((n + 1/2)F + G, P)) - (1 + \chi_F^{X, F+}(nF + G, P)) = (1 - \Lambda^4) - 1 + \Lambda^4 = 0$$

and using the result for $n = -1, r = 0$, the result for $r = 0$ follows by ascending and descending induction over $n \in \frac{1}{2}\mathbb{Z}$.

Let $r > 0, n \in \frac{1}{2}\mathbb{Z}$. By Proposition 7.3 we have

$$\begin{aligned} \chi_F^{X, F+}(nF + G, P^{2r+1}) - (1 - \Lambda^4)\chi_F^{X, F+}((n - 1/2)F + G, P^{2r-1}) \\ = \text{Coeff}_{q^0} \left[\frac{1}{2 \sinh(3h/2)} \tilde{\theta}_4(h)^{6n+9} M^{2r-1} (M^2 \tilde{\theta}_4(h)^3 - (1 - \Lambda^4)) u' h^* \Lambda^2 \right] \\ = \text{Coeff}_{q^0} [\tilde{\theta}_4(h)^{6n+9} g_3^r], \end{aligned}$$

where the last line is by Lemma 7.7(4). As $\tilde{\theta}_4(h)^{6n+9} \in \mathbb{Q}[[nq^2\Lambda^2, nq^4, q^2\Lambda^2, q^4]]^\times$, and g_3^r is a polynomial in $q^{-2}\Lambda^2, \Lambda^4$ of degree r , we find, as in the proof of Proposition 7.9, that

$$h'_r(n, \Lambda^4) := \chi_F^{X, F+}(nF + G, P^{2r+1}) - (1 - \Lambda^4)\chi_F^{X, F+}((n - 1/2)F + G, P^{2r-1}) \in \mathbb{Q}[\Lambda^4, n\Lambda^4]_{\leq r}.$$

Assume now by induction on r that

$$\chi_F^{X, F+}((n - 1/2)F + G, P^{2r-1}) = \frac{1}{1 - \Lambda^4}^{2n-2r+2} + h_{r-1}^1((n - 1/2), \Lambda^4)$$

with $h_{r-1}^1(n, \Lambda^4) \in \mathbb{Q}[\Lambda^4, n\Lambda^4]_{\leq r-1}$. Then

$$\chi_F^{X, F+}(nF + G, P^{2r+1}) - \frac{1}{(1 - \Lambda^4)^{2n-2r+1}} = (1 - \Lambda^4)h_{r-1}^1((n - 1/2), \Lambda^4) + h'_r(n, \Lambda^4).$$

Thus we put

$$h_r^1(n, \Lambda^4) := (1 - \Lambda^4)h_{r-1}^1((n - 1/2), \Lambda^4) + h'_r(n, \Lambda^4).$$

As $h'_r(n, \Lambda^4)$ has degree at most r in $\Lambda^4, n\Lambda^4$, the claim follows.

(2) The case $r = 0$ is proven in [10, Prop 5.16], with $\bar{h}_0^1(n, \Lambda^4) = -1 - (3n + 7)\Lambda^4$. For $r > 0$ we prove the result by induction. Let $r > 0$, then we have by Proposition 7.3 and Lemma 7.7(5)

$$\begin{aligned} \chi_0^{X, F+}(nF + G, P^{2r}) - (1 - \Lambda^4)\chi_0^{X, F+}((n - 1/2)F + G, P^{2r-2}) \\ = \text{Coeff}_{q^0} \left[-\frac{1}{2} \coth(3h/2) \tilde{\theta}_4(h)^{6n+9} M^{2r-2} (M^2 \tilde{\theta}_4(h)^3 - (1 - \Lambda^4)) u' h^* \Lambda^2 \right] \\ = \text{Coeff}_{q^0} [\tilde{\theta}_4(h)^{6n+9} g_4^r] =: l'_r(n, \Lambda^4) \in \mathbb{Q}[\Lambda^4, n\Lambda^4]_{\leq r+1}. \end{aligned}$$

Assume now that

$$\chi_0^{X, F+}((n - 1/2)F + G, P^{2r-2}) = \frac{1}{1 - \Lambda^4}^{2n-2r+3} + \bar{h}_{r-1}^1(n - 1/2, \Lambda^4),$$

with $\bar{h}_{r-1}^1(n-1/2, \Lambda^4) \in \mathbb{Q}[\Lambda^4, n\Lambda^4]_{\leq r}$. Then

$$\chi_0^{X, F+}(nF+G, P^{2r}) - \frac{1}{(1-\Lambda^4)^{2n-2r+2}} = (1-\Lambda^4)\bar{h}_{r-1}^1((n-1/2), \Lambda^4) + l'_r(n, \Lambda^4).$$

The result follows by induction on r . □

Remark 7.12. We list the $h_r^1(n, \Lambda^4)$, $\bar{h}_r^1(n, \Lambda^4)$ for small values of n ,

$$\begin{aligned} h_0^1 &= -1, & h_1^1 &= -1 + (6n+5)\Lambda^4 & h_2^1 &= -1 + (30n+19)\Lambda^4 - (18n^2+15n-2)\Lambda^8, \\ h_3^1 &= -1 + (126n+69)\Lambda^4 + (-162n^2+9n+114)\Lambda^8 + (36n^3-43n-7)\Lambda^{12}, \\ \bar{h}_0^1 &= -1 - (3n+7)\Lambda^4, & \bar{h}_1^1 &= -1 - (6n+20)\Lambda^4 + (9n^2+\frac{69}{2}n+32)\Lambda^8, \\ \bar{h}_2^1 &= -1 - (18n+78)\Lambda^4 + (54n^2+189n+120)\Lambda^8 - (18n^3+81n^2+109n+40)\Lambda^{12}. \end{aligned}$$

Proposition 7.13. *Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ or $X = \widehat{\mathbb{P}}^2$.*

(1) *For all $n \in \mathbb{Z}$*

$$\chi_F^{X, F+}(nF+2G) = \frac{1}{2} \frac{(1+\Lambda^4)^n - (1-\Lambda^4)^n}{(1-\Lambda^4)^{3n+3}}.$$

(2) *For all $n \in \mathbb{Z}$ and all $r > 0$ we have*

$$\chi_F^{X, F+}(nF+2G, P^{2r}) = \frac{2^{r-1}(1+\Lambda^4)^{n-r}}{(1-\Lambda^4)^{3n+3-2r}} - h_r^2(n, \Lambda^4),$$

where $h_r^2(n, \Lambda^4) \in \mathbb{Q}[\Lambda^4, n\Lambda^4]_{\leq 2r+2}$.

(3)

$$\chi_0^{X, F+}(nF+2G) = \frac{1}{2} \frac{(1+\Lambda^4)^n + (1-\Lambda^4)^n}{(1-\Lambda^4)^{3n+3}} - 1 - (4n+9)\Lambda^4.$$

(4) *For all $n \in \mathbb{Z}$ and all $r > 0$ we have*

$$\chi_0^{X, F+}(nF+2G, P^{2r}) = \frac{2^{r-1}(1+\Lambda^4)^{n-r}}{(1-\Lambda^4)^{3n+3-2r}} - \bar{h}_r^2(n, \Lambda^4),$$

where $\bar{h}_r^2(n, \Lambda^4) \in \mathbb{Q}[\Lambda^4, n\Lambda^4]_{\leq 2r+2}$.

Proof. (1) and (3) were proven in [10, Prop. 5.17]. (2) We will first show by induction on r that

$$(7.14) \quad -\frac{1}{2} \text{Coeff}_{q^0} [\tanh(h) \tilde{\theta}_4(h)^{8(n+2)} u' h^* \Lambda^2 M^{2r}] = 2^r \frac{(1+\Lambda^4)^{n-r}}{(1-\Lambda^4)^{3n+3-2r}} + s'_r(n, \Lambda^4).$$

For polynomials $s'_r(n, \Lambda^4) \in \mathbb{Q}[\Lambda^4, n\Lambda^4]_{\leq 2r+2}$ For $r = 0$ this is shown in the proof of [10, Prop. 5.17] with $s'_0 = -1 - (4n+9)\Lambda^4$.

Fix $r > 0$, assume that (7.14) holds $r-1$ and for all $n \in \mathbb{Z}$. By Lemma 7.7(6) we have

$$\begin{aligned} -\frac{1}{2} \text{Coeff}_{q^0} [\tanh(h) (M^{2r} \tilde{\theta}_4(h)^{8(n+2)} - (1-\Lambda^4)^2 M^{2r-2} \tilde{\theta}_4(h)^{8(n+2)} - \tilde{\theta}_4(h)^{8(n+1)} M^{2r-2}) u' h^* \Lambda^2] \\ = \text{Coeff}_{q^0} [\tilde{\theta}_4(h)^{8(n+1)} g_5^r] =: s''_r(n, \Lambda^4). \end{aligned}$$

Again, as $\tilde{\theta}_4(h) \in \mathbb{Q}[[\Lambda^2 q^4, q^4]]^\times$, and g_5^r has degree $r + 2$ in $q^{-2}\Lambda^2, \Lambda^4$, we see that $s_r''(n, \Lambda^4) \in \mathbb{Q}[\Lambda^4, n\Lambda^4]_{\leq r+2}$. Thus we get by induction on r

$$\begin{aligned} -\frac{1}{2} \text{Coeff}_{q^0} [\tanh(h) M^{2r} \tilde{\theta}_4(h)^{8(n+2)} u' h^* \Lambda^2] &= \frac{2^{r-1}(1 + \Lambda^4)^{n-r+1}}{(1 - \Lambda^4)^{3n+3-2r}} + (1 - \Lambda^4)^2 s_{r-1}'(n, \Lambda^4) \\ &+ \frac{2^{r-1}(1 + \Lambda^4)^{n-r}}{(1 - \Lambda^4)^{3n+2-2r}} + s_{r-1}'(n - 1, \Lambda^4) + s_r''(n, \Lambda^4) = \frac{2^r(1 + \Lambda^4)^{n-r}}{(1 - \Lambda^4)^{3n+3-2r}} + s_r'(n, \Lambda^4) \end{aligned}$$

with

$$s_r'(n, \Lambda^4) = (1 - \Lambda^4)^2 s_{r-1}'(n, \Lambda^4) + s_{r-1}'(n - 1, \Lambda^4) + s_r''(n, \Lambda^4).$$

As $s_{r-1}' \in \mathbb{Q}[\Lambda^4, n\Lambda^4]_{\leq 2r}$, $s_r'' \in \mathbb{Q}[\Lambda^4, n\Lambda^4]_{\leq r+2}$, we get $s_r'(n, \Lambda^4) \in \mathbb{Q}[\Lambda^4, n\Lambda^4]_{\leq 2r+2}$.

Now we show (2): We note that $\frac{1}{2\sinh(2h)} = \frac{1}{4}(\coth(h) - \tanh(h))$. Therefore we get by Proposition 7.3

$$\begin{aligned} \chi^{X, F+}(nF + 2G, P^{2r}) &= \frac{1}{4} \text{Coeff}_{q^0} [(\coth(h) - \tanh(h)) \tilde{\theta}_4(h)^{8(n+2)} M^{2r} u' h^* \Lambda^2] \\ &= -\frac{1}{2} \chi_0^{X, F+}((2n + 2)F, P^{2r}) + \frac{1}{2} \text{Coeff}_{q^0} [(-\frac{1}{2} \tanh(h)) \tilde{\theta}_4(h)^{8(n+2)} M^{2r} u' h^* \Lambda^2] \\ &= -\frac{1}{2} \bar{h}_r^0(2n + 2, \Lambda^4) + \frac{2^{r-1}(1 + \Lambda^4)^{n-r}}{(1 - \Lambda^4)^{3n+3-2r}} + \frac{1}{2} s_r'(n, \Lambda^4). \end{aligned}$$

here in the last line we have used Proposition 7.9 and (7.14). The claim follows with $h_r^2(n, \Lambda^4) = \frac{1}{2}(s_r'(n, \Lambda^4) - \bar{h}_r^0(2n + 2, \Lambda^4)) \in \mathbb{Q}[\Lambda^4, n\Lambda^4]_{\leq 2r+2}$.

Finally we show (4): $-\frac{1}{2} \coth(2h) = \frac{1}{4}(-\coth(h) - \tanh(h))$ and Proposition 7.3 give

$$\begin{aligned} \chi^{X, F+}(nF + 2G, P^{2r}) &= \frac{1}{4} \text{Coeff}_{q^0} [(-\coth(h) - \tanh(h)) \tilde{\theta}_4(h)^{8(n+2)} M^{2r} u' h^* \Lambda^2] \\ &= \frac{1}{2} \chi_0^{X, F+}((2n + 2)F, P^{2r}) + \frac{1}{2} \text{Coeff}_{q^0} [(-\frac{1}{2} \tanh(h)) \tilde{\theta}_4(h)^{8(n+2)} M^{2r} u' h^* \Lambda^2] \\ &= \frac{1}{2} \bar{h}_r^0(2n + 2, \Lambda^4) + \frac{2^{r-1}(1 + \Lambda^4)^{n-r}}{(1 - \Lambda^4)^{3n+3-2r}} + \frac{1}{2} s_r'(n, \Lambda^4). \end{aligned}$$

The claim follows with $\bar{h}_r^2 = \frac{1}{2}(s_r'(n, \Lambda^4) + \bar{h}_r^0(2n + 2, \Lambda^4))$. □

Remark 7.15. Again we can readily compute the first few of the h_r^2, \bar{h}_r^2 .

$$\begin{aligned} h_1^2 &= -1, \quad h_2^2 = -2 + (8n + 6)\Lambda^4, \quad h_3^2 = -4 + (48n + 24)\Lambda^4 - (32n^2 + 28n)\Lambda^8, \\ h_4^2 &= -8 + (224n + 72)\Lambda^4 + (-320n^2 - 40n + 128)\Lambda^8 + (\frac{256}{3}n^3 + 32n^2 - \frac{184}{3}n - 16)\Lambda^{12}, \\ \bar{h}_1^2 &= -1 - (8n + 24)\Lambda^4 + (16n^2 + 62n + 59)\Lambda^8, \\ \bar{h}_2^2 &= -2 - (24n + 90)\Lambda^4 + (96n^2 + 348n + 270)\Lambda^8 - (\frac{128}{3}n^3 + 208n^2 + \frac{964}{3}n + 154)\Lambda^{12}. \end{aligned}$$

It appears that one has $h_r^2 \in \mathbb{Q}[\Lambda^4, n\Lambda^4]_{\leq r-1}$ and $\bar{h}_r^2 \in \mathbb{Q}[\Lambda^4, n\Lambda^4]_{\leq r}$.

Corollary 7.16. *Let $X = \widehat{\mathbb{P}}^2$ or $X = \mathbb{P}^1 \times \mathbb{P}^1$, let $\omega \in H^2(X, \mathbb{R})$ be a class with $\langle \omega^2 \rangle > 0$.*

(1) For $n \in \mathbb{Z}_{\geq 0}$ we have

$$\chi_0^{X,\omega}(nF) \equiv \chi_F^{X,\omega}(nF) \equiv \frac{1}{(1-\Lambda^4)^{n+1}}, \quad \chi_0^{X,\omega}(nF, P^{2r}) \equiv \chi_F^{X,\omega}(nF, P^{2r}) \equiv 0 \text{ for } r > 0.$$

(2) For $n \in \mathbb{Z}_{\geq 0}$ if $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $n \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$ if $X = \widehat{\mathbb{P}}^2$, we have

$$\chi_0^{X,\omega}(nF + G, P^{2r}) \equiv \frac{1}{(1-\Lambda^4)^{2n+2-2r}}, \quad \chi_F^{X,\omega}(nF + G, P^{2r+1}) \equiv \frac{1}{(1-\Lambda^4)^{2n+1-2r}}.$$

(3) For $n \in \mathbb{Z}_{\geq 0}$ we have

$$\begin{aligned} \chi_0^{X,\omega}(nF + 2G) &\equiv \frac{(1+\Lambda^4)^n + (1-\Lambda^4)^n}{2(1-\Lambda^4)^{3n+3}}, & \chi_F^{X,\omega}(nF + 2G) &\equiv \frac{(1+\Lambda^4)^n - (1-\Lambda^4)^n}{2(1-\Lambda^4)^{3n+3}} \\ \chi_0^{X,\omega}(nF + 2G, P^{2r}) &\equiv \chi_F^{X,\omega}(nF + 2G, P^{2r}) \equiv \frac{2^{r-1}(1+\Lambda^4)^{n-r}}{(1-\Lambda^4)^{3n+3-2r}} \text{ for } 1 \leq r \leq n. \end{aligned}$$

Proof. Write $\omega = uF + vG$, with $u, v \in \mathbb{R}_{>0}$, write $w = v/u$. By Proposition 7.9, Proposition 7.11, Proposition 7.13 and Lemma 4.15 it is sufficient to prove that for $L = nF + mG$, with $0 \leq m \leq 2$, under the assumptions of the Corollary, there are only finitely many classes ξ of type (0) or type (F) with $\langle \omega, \xi \rangle \geq 0 > \langle F, \xi \rangle$, such that $\delta_\xi^X(nF + mG, P^s) \neq 0$, i.e. such that $-\xi^2 \leq |\langle \xi, (L - K_X) \rangle| + s + 2$. These walls are of the form $\xi = aF - bG$ with $a \in \mathbb{Z}_{>0}$ and $b \in 2\mathbb{Z}_{>0}$, and $aw \geq b$, and the condition becomes

$$(7.17) \quad 2ab \leq |b(n+2) - a(m+2)| + s + 2.$$

Let $\xi = (aF - bG)$ be such a wall with $\delta_\xi^X(nF + mG, P^s) \neq 0$. If $a(m+2) \leq b(n+2)$, then (7.17) becomes

$$2ab \leq b(n+2) - a(m+2) + s + 2 \leq b(n+2) + s,$$

therefore $(2a - n - 2)b \leq s$. Therefore $2a - n - 2 \leq \frac{s}{b} \leq \frac{s}{2}$. Therefore a is bounded, and by the condition $b \leq aw$ also b is bounded, so there are only finitely many possibilities for a, b .

Now assume $a(m+2) \geq b(n+2)$. Then as $b \geq 2$, (7.17) gives

$$4a \leq 2ab \leq a(m+2) - 2(n+2) + s + 2,$$

i.e. $(2-m)a \leq -2n + s - 2$. If $m = 0, 1$, then $a \leq \frac{-2n+s-2}{2-m}$, thus a is bounded, and by $a(m+2) \geq b(n+2)$ also b is bounded. If $m = 2$, the inequality becomes $2n \leq s - 2$, so if $2n \geq s$ there are no walls with $\delta_\xi^X(nF + mG, P^s) \neq 0$. Thus the claim follows. \square

Remark 7.18. As we will use this later in §8.4, we explicitly state the bounds obtained in the above proof in the case of $X = \widehat{\mathbb{P}}^2$, $\omega = H$ (i.e. $w = 2$ in the notation above). Fix $n \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 0}$. Let $\xi = aF - bG$ be a class of type (0) or (F) with $\langle \xi, \omega \rangle \geq 0 > \langle \xi, F \rangle$.

(1) If $\delta_\xi^X(nF - nE, P^s) = \delta_\xi^X(nF, P^s) \neq \emptyset$, then

- (a) either $2a \leq (n+2)b$ and $0 < a \leq \frac{n+2}{2} + \frac{s}{4}$ and $0 < b \leq 2a$,
- (b) or $0 < (n+2)b \leq 2a$ and $0 < a \leq \frac{s}{2} - n - 1$.

(2) If $\delta_\xi^X(nF - (n-1)E, P^s) = \delta_\xi^X((n-1/2)F + G, P^s) \neq \emptyset$, then

- (a) either $3a \leq (n + \frac{3}{2})b$ and $0 < a \leq \frac{n+3/2}{2} + \frac{s}{4}$ and $0 < b \leq 2a$,

(b) or $0 < (n + 3/2)b \leq 3a$ and $0 < a \leq s - 2n - 2$.

Remark 7.19. Note that the results of Corollary 7.16 are compatible with Conjecture 1.7. This is particularly remarkable for part (3) of Corollary 7.16, which can only be proven for $r \leq n$, while its correctness for $r > n$ would contradict Conjecture 1.7.

The fact that the formulas hold without restriction for $\chi_0^{X,F+}(nF + 2G, P^{2r})$, $\chi_F^{X,F+}(nF + 2G, P^{2r})$ is not in contradiction to Conjecture 1.7, because it is only claimed for $\chi_{c1}^{Y,\omega}(L, P^r)$ with ω an ample class on Y .

8. COMPUTATION OF THE INVARIANTS OF THE PLANE

We now want to use the results obtained so far to give an algorithm to compute the generating functions $\chi_0^{\mathbb{P}^2,H}(nH, P^r)$, $\chi_H^{\mathbb{P}^2,H}(nH, P^r)$ of the K -theoretic Donaldson invariants of the projective plane. We use this algorithm to prove that these generating functions are always rational functions of a very special kind. Then we will use this algorithm to explicitly compute $\chi_0^{\mathbb{P}^2,H}(nH, P^r)$, $\chi_H^{\mathbb{P}^2,H}(nH, P^r)$ for not too large values of n and r . First we explicitly carry out the algorithm by hand when $r = 0$ and $n \leq 5$ in an elementary but tedious computation. Finally we implemented the algorithm as a PARI program, which in principle can prove a formula for $\chi_0^{\mathbb{P}^2,H}(nH, P^r)$, $\chi_H^{\mathbb{P}^2,H}(nH, P^r)$ for any n and r . The computations have been carried out for $r = 0$ and $n \leq 11$ and $n \leq 8$ and $r \leq 16$.

8.1. The strategy. Corollary 6.17 says in particular the following.

Remark 8.1. (1) For all $n \in \mathbb{Z}_{>0}$ there exist unique polynomials $f_n, g_n \in \mathbb{Q}[x, \lambda^4]$ and an integer N_n , such that $f_n S_n + g_n S_{n+1} = \lambda(1 - \lambda^4)^{N_n}$.
 (2) For all $n \in \mathbb{Z}_{>0}$ there exist unique polynomials $h_n, l_n \in \mathbb{Q}[x^2, \lambda^4]$ and an integer M_n , such that $h_n R_n + l_n R_{n+1} = (1 - \lambda^4)^{M_n}$.

Using these polynomials, we can determine the K -theoretic Donaldson invariants of \mathbb{P}^2 in terms of those of $\widehat{\mathbb{P}^2}$.

Corollary 8.2. *For all $n, k \in \mathbb{Z}$, $r \in \mathbb{Z}_{\geq 0}$ we have*

$$\begin{aligned} \text{(H)} \quad \chi_H^{\mathbb{P}^2,H}(nH, P^r) &= \frac{1}{\Lambda(1 - \Lambda^4)^{N_k}} \left(\chi_F^{\widehat{\mathbb{P}^2},H} \left((n+1-k)G + \frac{n+k-1}{2}F, P^r \cdot f_k(P, \Lambda) \right) \right. \\ &\quad \left. + \chi_F^{\widehat{\mathbb{P}^2},H} \left((n-k)G + \frac{n+k}{2}F, P^r \cdot g_k(P, \Lambda) \right) \right), \\ \text{(0)} \quad \chi_0^{\mathbb{P}^2,H}(nH, P^r) &= \frac{1}{(1 - \Lambda^4)^{M_k}} \left(\chi_0^{\widehat{\mathbb{P}^2},H} \left((n+1-k)G + \frac{n+k-1}{2}F, P^r \cdot h_k(P, \Lambda) \right) \right. \\ &\quad \left. + \chi_0^{\widehat{\mathbb{P}^2},H} \left((n-k)G + \frac{n+k}{2}F, P^r \cdot l_k(P, \Lambda) \right) \right). \end{aligned}$$

Proof. Note that $(n-k)G + \frac{n+k}{2}F = nH - kE$. Therefore we get by Theorem 6.7 that

$$\chi_F^{\widehat{\mathbb{P}^2},H} \left((n+1-k)G + \frac{n+k-1}{2}F, P^r \cdot f_k(P, \Lambda) \right) + \chi_F^{\widehat{\mathbb{P}^2},H} \left((n-k)G + \frac{n+k}{2}F, P^r \cdot g_k(P, \Lambda) \right)$$

$$= \chi_H^{\mathbb{P}^2, H}(nH, P^r \cdot (f_k(P, \Lambda)S_k(P, \Lambda) + g_k(P, \Lambda)S_{k+1}(P, \Lambda))),$$

and the result follows by $f_k(P, \Lambda)S_k(P, \Lambda) + g_k(P, \Lambda)S_{k+1}(P, \Lambda) = \Lambda(1 - \Lambda^4)^{N_k}$. In the same way

$$\begin{aligned} \chi_0^{\widehat{\mathbb{P}}^2, H} \left((n+1-k)G + \frac{n+k-1}{2}F, P^r \cdot h_k(P, \Lambda) \right) &+ \chi_0^{\widehat{\mathbb{P}}^2, H} \left((n-k)G + \frac{n+k}{2}F, P^r \cdot l_k(P, \Lambda) \right) \\ &= \chi_0^{\mathbb{P}^2, H}(nH, P^r \cdot (h_k(P, \Lambda)R_k(P, \Lambda) + l_k(P, \Lambda)R_{k+1}(P, \Lambda))), \end{aligned}$$

and $h_k(P, \Lambda)R_k(P, \Lambda) + l_k(P, \Lambda)R_{k+1}(P, \Lambda) = (1 - \Lambda^4)^{M_k}$. \square

Using Corollary 7.16 we can use this in two different ways to compute the K -theoretic Donaldson invariants of \mathbb{P}^2 .

- (1) We apply parts (1) and (2) of Corollary 7.16 to compute the $\chi_0^{\widehat{\mathbb{P}}^2, H}(nF, P^s)$, $\chi_F^{\widehat{\mathbb{P}}^2, H}(nF, P^s)$, $\chi_0^{\widehat{\mathbb{P}}^2, H}(G + (n - \frac{1}{2})F, P^s)$, $\chi_F^{\widehat{\mathbb{P}}^2, H}(G + (n - \frac{1}{2})F, P^s)$, and then apply Corollary 8.2 with $k = n$. Parts (1) and (2) of Corollary 7.16 apply for all values of n and s , so this method can always be used. We will apply this §8.2 to prove the rationality of the generating functions of the K -theoretic Donaldson invariants of \mathbb{P}^2 and of blowups of \mathbb{P}^2 , and then in §8.4 to compute the K -theoretic Donaldson invariants of \mathbb{P}^2 using a PARI program.
- (2) We apply parts (2) and (3) of Corollary 7.16 to compute the $\chi_0^{\widehat{\mathbb{P}}^2, H}(G + (n - \frac{1}{2})F, P^s)$, $\chi_F^{\widehat{\mathbb{P}}^2, H}(G + (n - \frac{1}{2})F, P^s)$, $\chi_0^{\widehat{\mathbb{P}}^2, H}(2G + (n - 1)F, P^s)$, $\chi_F^{\widehat{\mathbb{P}}^2, H}(2G + (n - 1)F, P^s)$, and then apply Corollary 8.2 with $k = n - 1$. This requires less computation than the first approach. However, as part (3) of Corollary 7.16 holds for $\chi_0^{\widehat{\mathbb{P}}^2, H}(2G + (n - 1)F, P^s)$, $\chi_F^{\widehat{\mathbb{P}}^2, H}(2G + (n - 1)F, P^s)$ only when $s \leq 2n - 2$, this method only allows to compute $\chi_0^{\mathbb{P}^2, H}(nH, P^r)$ when

$$r + \max(\deg_x(h_{n-1}(x, \lambda), \deg_x(l_{n-1}(x, \lambda))) \leq 2n - 2,$$

and the same way it only allows to compute $\chi_H^{\mathbb{P}^2, H}(nH, P^r)$ when

$$r + \max(\deg_x(f_{n-1}(x, \lambda), \deg_x(g_{n-1}(x, \lambda))) \leq 2n - 2.$$

As the degree of S_n and R_n in x grows faster than $2n$, this requires that n and r are both relatively small. We will use this to compute $\chi_0^{\mathbb{P}^2, H}(nH)$, $\chi_H^{\mathbb{P}^2, H}(nH)$ by hand for $n = 4, 5$.

8.2. Rationality of the generating function. We now use the above algorithm to prove a structural result about the K -theoretic Donaldson invariants of \mathbb{P}^2 and the blowups of \mathbb{P}^2 .

Theorem 8.3. (1) *For all $n \in \mathbb{Z}$, $r \in \mathbb{Z}_{\geq 0}$ with $n + r$ even, there exists an integer $d_{n,r}^1$ and a polynomial $p_{n,r}^1 \in \mathbb{Q}[\Lambda^4]$, such that*

$$\chi_H^{\mathbb{P}^2, H}(nH, P^r) \equiv \frac{p_{n,r}^1}{\Lambda(1 - \Lambda^4)^{d_{n,r}^1}}.$$

Furthermore we can choose $p_{n,0}^1 \in \mathbb{Z}[\Lambda^4]$.

- (2) For all $n \in \mathbb{Z}$, $r \in 2\mathbb{Z}_{\geq 0}$ there exists an integer $d_{n,r}^0$ and a polynomial $p_{n,r}^0 \in \mathbb{Q}[\Lambda^4]$, such that

$$\chi_0^{\mathbb{P}^2, H}(nH, P^r) \equiv \frac{p_{n,r}^0}{(1 - \Lambda^4)^{d_{n,r}^0}}.$$

Furthermore we can choose $p_{n,0}^0 \in \mathbb{Z}[\Lambda^4]$.

Proof. By Corollary 7.16 there exist for all $L = nF$, $L = (n - 1/2)F + G$ with $n \in \mathbb{Z}$ and all $r \in \mathbb{Z}_{\geq 0}$ integers $e_{n,r}^F$, $e_{n,r}^0$ and polynomials $q_{n,r}^F$, $q_{n,r}^0 \in \mathbb{Q}[\Lambda^4]$, so that

$$\chi_F^{\widehat{\mathbb{P}}^2, H}(L, P^r) = \frac{q_{n,r}^F}{(1 - \Lambda^4)^{e_{n,r}^F}}, \quad \chi_0^{\widehat{\mathbb{P}}^2, H}(L, P^r) = \frac{q_{n,r}^0}{(1 - \Lambda^4)^{e_{n,r}^0}}.$$

Thus part (H) of Corollary 8.2 (with $k = n$) gives $\chi_H^{\mathbb{P}^2, H}(nH, P^r) = \frac{p_{n,r}^1}{\Lambda(1 - \Lambda^4)^{d_{n,r}^1}}$, for suitable $d_{n,r}^1 \in \mathbb{Z}_{\geq 0}$, $p_{n,r}^1 \in \mathbb{Q}[\Lambda^4]$ and similarly part (0) of Corollary 8.2 (with $k = n$) gives $\chi_0^{\mathbb{P}^2, H}(nH, P^r) = \frac{p_{n,r}^0}{(1 - \Lambda^4)^{d_{n,r}^0}}$, for suitable $d_{n,r}^0 \in \mathbb{Z}_{\geq 0}$, $p_{n,r}^0 \in \mathbb{Q}[\Lambda^4]$. Finally we want to see that $p_{n,0}^1 \in \mathbb{Z}[\Lambda^4]$, and we can choose $p_{n,0}^0$ so that it is in $\mathbb{Z}[\Lambda^4]$. By definition we have

$$\chi_H^{\mathbb{P}^2, H}(nH) = \sum_{k > 0} \chi(M_H^X(H, 4k - 1), \mu(nH)) \Lambda^k \in \Lambda^3 \mathbb{Z}[[\Lambda^4]].$$

Writing $p_{n,0}^0 = \sum_{k > 0} a_k \Lambda^{4k}$ we see from the formula $\chi_H^{\mathbb{P}^2, H}(nH) = \frac{p_{n,0}^1}{\Lambda(1 - \Lambda^4)^{d_{n,0}^1}}$ that $a_0 = 0$, and inductively that

$$a_k = \chi(M_H^X(H, 4k - 1)) - \sum_{i=1}^k a_{k-i} \binom{d_{n,0}^1 + i - 1}{i} \in \mathbb{Z}.$$

For k large enough we have that the coefficient of Λ^{4k} of $\chi_0^{\mathbb{P}^2, H}(nH)$ is $\chi(M_H^X(0, 4k), \mu(nH))$. Thus, adding a polynomial $h \in \mathbb{Q}[\Lambda^4]$ to $p_{n,0}^0$, we can assume that $\frac{p_{n,0}^0}{(1 - \Lambda^4)^{d_{n,0}^0}} \in \mathbb{Z}[\Lambda^4]$. One concludes in the same way as for $p_{n,0}^1$. \square

Indeed a more careful argument will show that we can choose $p_{n,r}^0$, $p_{n,r}^1 \in \mathbb{Z}[\Lambda^4]$ for all r .

We now use this result and the blowup formulas to describe the generating functions of the K -theoretic Donaldson invariants of blowups of \mathbb{P}^2 in finitely many points in an open subset of the ample cone as rational functions.

Lemma 8.4. *Let X be the blowup of \mathbb{P}^2 in finitely many points, p_1, \dots, p_n , and denote by E_1, \dots, E_n the exceptional divisors. Fix $c_1 \in H^2(X, \mathbb{Z})$, and an $r \geq 0$. Let L be a line bundle on X . Let $\omega = H - \alpha_1 E_1 - \dots - \alpha_n E_n$ with $|\alpha_i| < \frac{1}{\sqrt{n}}$, for all i , and $\langle \omega, K_X \rangle < 0$. Then $\chi_{c_1}^{X, \omega}(L, P^r) \equiv \chi_{c_1}^{X, H}(L, P^r)$.*

Proof. We put $\epsilon := \max(|\alpha_i|)_{i=1}^n$, and $\delta := \frac{1}{n} - \epsilon^2 > 0$. Let $L = dH - m_1 E_1 - \dots - m_n E_n$, with $d, m_1, \dots, m_n \in \mathbb{Z}$, and let $r \geq 0$. We want to show that there are only finitely many classes ξ of type (c_1) on X with $\langle H, \xi \rangle \geq 0 \geq \langle \omega, \xi \rangle$ and $\delta_\xi^X(L, P^r) \neq 0$. As by Lemma 4.15 each $\delta_\xi^X(L, P^r)$ is a polynomial in Λ , this gives $\chi_{c_1}^{X, \omega}(L, P^r) \equiv \chi_{c_1}^{X, H}(L, P^r)$.

We write $\xi = aH - b_1E_1 - \dots - b_nE_n$, and $b := (|b_1| + \dots + |b_n|)$; then we get $a \geq 0$ and

$$0 \geq \langle \omega, \xi \rangle = a - \alpha_1 b_1 - \dots - \alpha_n b_n \geq a - b\epsilon,$$

i.e. $a \leq b\epsilon$. Assume $\delta_\xi^X(L, P^r) \neq 0$, then by Lemma 4.15 $-\xi^2 \leq |\langle \xi, (L - K_X) \rangle| + r + 2$. We have

$$(8.5) \quad \xi^2 = -a^2 + b_1^2 + \dots + b_n^2 \geq -\epsilon^2 b^2 + \frac{b^2}{n} = \delta b^2,$$

where we have used the easy inequality $b_1^2 + \dots + b_n^2 \geq \frac{b^2}{n}$ and our definition $\frac{1}{n} - \epsilon^2 = \delta > 0$.

On the other hand, putting $m := \max |m_i + 1|_{i=1}^n$ we get

$$\begin{aligned} |\langle \xi, (L - K_X) \rangle| + r + 2 &= |a(d+3) - (m_1+1)b_1 - \dots - (m_n+1)b_n| + r + 2 \\ &\leq a|d+3| + |m_1+1||b_1| + \dots + |m_n+1||b_n| + r + 2 \\ &\leq \epsilon b|d+3| + mb + r + 2 = (m + |d+3|\epsilon)b + r + 2. \end{aligned}$$

Putting this together with (8.5), and using $\epsilon \leq 1$, we get

$$(8.6) \quad \delta(|b_1| + \dots + |b_n|) \leq \max |m_i + 1|_{i=1}^n + |d+3| + \frac{r+2}{|b_1| + \dots + |b_n|}.$$

thus $b = |b_1| + \dots + |b_n|$ is bounded and $a \leq b\epsilon$ is bounded, and therefore there are only finitely many choices for ξ . \square

The following theorem contains Theorem 1.2 as a special case.

Theorem 8.7. *Let X be the blowup of \mathbb{P}^2 in finitely many points. With the assumptions and notations of Lemma 8.4, there exist an integer $d_{L,r}^{c_1} \in \mathbb{Z}_{\geq 0}$ and a polynomial $p_{L,r}^{c_1} \in \mathbb{Q}[\Lambda^{\pm 4}]$, such that*

$$\chi_{c_1}^{X,\omega}(L, P^r) \equiv \frac{p_{L,r}^{c_1}}{\Lambda^{c_1^2}(1 - \Lambda^4)^{d_{L,r}^{c_1}}}.$$

Proof. We write $c_1 = kH + l_1E_1 + \dots + l_nE_n$. By renumbering the E_i we can assume that l_i is odd for $1 \leq i \leq s$ and l_i is even for $s+1 \leq i \leq n$. Write $L = dH - m_1E_1 - \dots - m_nE_n$, with $d, m_1, \dots, m_n \in \mathbb{Z}$.

By Lemma 8.4, it is enough to show the claim for $\omega = H$. By repeatedly applying Theorem 6.7, we get

$$\chi_{c_1}^{X,H}(L, P^r) = \chi_{kH}^{\mathbb{P}^2,H} \left(dH, P^r \cdot \left(\prod_{i=1}^s S_{m_i+1}(P, \Lambda) \right) \cdot \left(\prod_{i=s+1}^n R_{m_i+1}(P, \Lambda) \right) \right).$$

Put $\kappa = 0$ if k is even, and $\kappa = 1$ if k is odd. We know that $\chi_{kH}^{\mathbb{P}^2,H}(dH, P^r)$ depends only on κ , and by Theorem 8.3 we have

$$\chi_{kH}^{\mathbb{P}^2,H}(dH, P^r) = \frac{p_{d,r}^\kappa}{\Lambda^\kappa(1 - \Lambda^4)^{d_{d,r}^\kappa}},$$

We know that $R_n(P, \Lambda) \in \mathbb{Z}[P, \Lambda^4]$, $S_n(P, \Lambda) \in \Lambda\mathbb{Z}[P, \Lambda^4]$. Therefore we can write $\chi_{c_1}^{X,H}(L, P^r) = \frac{p}{\Lambda^{\kappa-s}(1-\Lambda^4)^N}$ for a suitable polynomial $p \in \mathbb{Q}[\Lambda^{\pm 4}]$ and a nonnegative integer N . Note that $c_1^2 = k^2 - l_1^2 - \dots - l_n^2 \equiv \kappa - s \pmod{4}$. Let $w := \frac{1}{4}(c_1^2 - (\kappa - s))$. Then

$$\chi_{c_1}^{X,H}(L, P^r) = \frac{p}{\Lambda^{\kappa-s}(1-\Lambda^4)^N} = \frac{\Lambda^{4w}p}{\Lambda^{c_1^2}(1-\Lambda^4)^N},$$

and the claim follows. \square

8.3. Explicit computations for small n . We compute $\chi_0^{\mathbb{P}^2,H}(nH)$, $\chi_H^{\mathbb{P}^2,H}(nH)$ for small values of n , using the blowup formulas, using the strategy outlined in §8.1. In the next subsection we do the same computations for larger n using a computer program written in Pari. These invariants have been computed before (see [1],[10]) for $1 \leq n \leq 3$.

Proposition 8.8. (1) $\chi_H^{\mathbb{P}^2,H}(4H) = \frac{\Lambda^3 + 6\Lambda^7 + \Lambda^{15}}{(1-\Lambda^4)^{15}}.$

$$(2) \chi_0^{\mathbb{P}^2,H}(4H) = \frac{1 + 6\Lambda^8 + \Lambda^{12}}{(1-\Lambda^4)^{15}} - 1 - 51/2\Lambda^4.$$

Proof. (1) We have

$$S_3(x, \lambda) = \lambda(x^2 - (1 - \lambda^4)^2), \quad S_4 = \lambda x((1 - \lambda^8)x^2 - 2(1 - \lambda^4)^3).$$

Using division with rest as polynomials in x , we write $\lambda(1 - \lambda^4)^6$ as a linear combination of $S_3(x, \lambda)$ and $S_4(x, \lambda)$

$$\lambda(1 - \lambda^4)^6 = ((1 - \lambda^8)x^2 - (1 - \lambda^4)^4) S_3(x, \lambda) - x S_4(x, \lambda).$$

Thus we get by Corollary 8.2 that

$$(8.9) \quad \begin{aligned} \chi_H^{\mathbb{P}^2,H}(4H) &= \frac{1}{\Lambda(1-\Lambda^4)^6} ((1-\Lambda^8)\chi_F^{\widehat{\mathbb{P}}^2,H}(4H-2E, P^2) - (1-\Lambda^4)^4\chi_F^{\widehat{\mathbb{P}}^2,H}(4H-2E) \\ &\quad - \chi_F^{\widehat{\mathbb{P}}^2,H}(4H-3E, P^1)). \end{aligned}$$

By Proposition 7.11 we have

$$\chi_F^{\widehat{\mathbb{P}}^2,F_+}(4H-3E, P^1) = \frac{1}{(1-\Lambda^4)^8} - 1,$$

and $\xi = H - 3E$ is the only class of type (F) on $\widehat{\mathbb{P}}^2$ with $\langle \xi, H \rangle \geq 0 > \langle \xi, F \rangle$ with $\delta_{\xi}^{\widehat{\mathbb{P}}^2}((4H - 3E), P^1) \neq 0$. In fact $\delta_{H-3E}^{\widehat{\mathbb{P}}^2}((4H - 3E), P^1) = \Lambda^8$. Thus

$$\chi_F^{\widehat{\mathbb{P}}^2,H}(4H-3E, P^1) = \frac{1}{(1-\Lambda^4)^8} - 1 + \Lambda^8.$$

By Proposition 7.13 we have that

$$\chi_F^{\widehat{\mathbb{P}}^2,F_+}(4H-2E) = \frac{3\Lambda^4 + \Lambda^{12}}{(1-\Lambda^4)^{12}}, \quad \chi_F^{\widehat{\mathbb{P}}^2,F_+}(4H-2E, P^2) = \frac{(1+\Lambda^4)^2}{(1-\Lambda^4)^{10}} - 1.$$

Furthermore there is no class of type (F) on $\widehat{\mathbb{P}}^2$ with $\langle \xi, H \rangle \geq 0 > \langle \xi, F \rangle$ with $\delta_{\xi}^{\widehat{\mathbb{P}}^2}(4H-2E) \neq 0$ or $\delta_{\xi}^{\widehat{\mathbb{P}}^2}(4H-2E, P^2) \neq 0$. Thus $\chi_F^{\widehat{\mathbb{P}}^2, H}(4H-2E) = \chi_F^{\widehat{\mathbb{P}}^2, F_+}(4H-2E)$ and $\chi_F^{\widehat{\mathbb{P}}^2, H}(4H-2E, P^2) = \chi_F^{\widehat{\mathbb{P}}^2, F_+}(4H-2E, P^2)$. Putting these values into (8.9) yields $\chi_H^{\widehat{\mathbb{P}}^2, H}(4H) = \frac{\Lambda^3 + 6\Lambda^7 + \Lambda^{15}}{(1-\Lambda^4)^{15}}$.

(2) For $R_3(x, \lambda) = -\lambda^4 x^2 + (1 - \lambda^4)^2$, $R_4 = -\lambda^4 x^2 + (1 - \lambda^4)^4$, we get

$$(1 - \lambda^4)^5 = (\lambda^4 x^2 + (1 - \lambda^4)^2) R_3(x, \lambda) - \lambda^4 R_4(x, \lambda).$$

Thus Corollary 8.2 gives

$$(8.10) \quad \chi_0^{\widehat{\mathbb{P}}^2, H}(4H) = \frac{1}{(1 - \Lambda^4)^5} \left(\Lambda^4 \chi_0^{\widehat{\mathbb{P}}^2, H}(4H - 2E, P^2) + (1 - \Lambda^4)^2 \chi_0^{\widehat{\mathbb{P}}^2, H}(4H - 2E) - \Lambda^4 \chi_0^{\widehat{\mathbb{P}}^2, H}(4H - 3E) \right).$$

By Proposition 7.11 we have $\chi_0^{\widehat{\mathbb{P}}^2, F_+}(4H - 3E) = \frac{1}{(1 - \Lambda^4)^9} - 1 - 35/2\Lambda^4$. Furthermore there are no classes ξ of type (0) with $\langle \xi, H \rangle > 0 > \langle \xi, F \rangle$ with $\delta_{\xi}^{\widehat{\mathbb{P}}^2}((4H - 3E)) \neq 0$, and the only classes of type (0) with $\langle \xi, H \rangle = 0 > \langle \xi, F \rangle$ are $-2E$ and $-4E$ with

$$\frac{1}{2} \delta_{-2E}^{\widehat{\mathbb{P}}^2}(4H - 3E) = -2\Lambda^4 + 291\Lambda^8 - 3531\Lambda^{12} + 16215/2\Lambda^{16}, \quad \frac{1}{2} \delta_{-4E}^{\widehat{\mathbb{P}}^2}(4H - 3E) = 7\Lambda^{16} - 51/2\Lambda^{20},$$

giving

$$\chi_0^{\widehat{\mathbb{P}}^2, H}(4H - 3E) = \frac{1}{(1 - \Lambda^4)^9} - 1 - 39/2\Lambda^4 + 291\Lambda^8 - 3531\Lambda^{12} + 16229/2\Lambda^{16} - 51/2\Lambda^{20}.$$

By Proposition 7.13 we have $\chi_0^{\widehat{\mathbb{P}}^2, F_+}(4H - 2E) = \frac{1+3\Lambda^8}{(1-\Lambda^4)^{12}} - 1 - 21\Lambda^4$. Furthermore the only class ξ of type (0) with $\langle \xi, H \rangle \geq 0 > \langle \xi, F \rangle$ with $\delta_{\xi}^{\widehat{\mathbb{P}}^2}(4H - 2E) \neq 0$ is $-2E$ with $\frac{1}{2} \delta_{-2E}^{\widehat{\mathbb{P}}^2}(4H - 2E) = -3/2\Lambda^4 + 108\Lambda^8 - 1225/2\Lambda^{12}$, giving

$$\chi_0^{\widehat{\mathbb{P}}^2, H}(4H - 2E) = \frac{1 + 3\Lambda^8}{(1 - \Lambda^4)^{12}} - 1 - 45/2\Lambda^4 + 108\Lambda^8 - 1225/2\Lambda^{12}.$$

By Proposition 7.13 we have $\chi_0^{\widehat{\mathbb{P}}^2, F_+}(4H - 2E, P^2) = \frac{(1+\Lambda^4)^2}{(1-\Lambda^4)^{10}} - 1 - 48\Lambda^4 + 389\Lambda^8$, and the classes ξ of type (0) with $\langle \xi, H \rangle \geq 0 > \langle \xi, F \rangle$ with $\delta_{\xi}^{\widehat{\mathbb{P}}^2}(4H - 2E, P^2) \neq 0$ are $-2E$ and $-4E$ with $\frac{1}{2} \delta_{-2E}^{\widehat{\mathbb{P}}^2}(4H - 2E, P^2) = -6\Lambda^4 + 508\Lambda^8 - 4614\Lambda^{12} + 8600\Lambda^{16}$ and $\frac{1}{2} \delta_{-4E}^{\widehat{\mathbb{P}}^2}(4H - 2E, P^2) = 1/2\Lambda^{16}$, giving

$$\chi_0^{\widehat{\mathbb{P}}^2, H}(4H - 2E) = \frac{(1 + \Lambda^4)^2}{(1 - \Lambda^4)^{10}} - 1 - 54\Lambda^4 + 897\Lambda^8 - 4614\Lambda^{12} + 17201/2\Lambda^{16}.$$

Putting this into (8.10) gives $\chi_0^{\widehat{\mathbb{P}}^2, H}(4H) = \frac{1+6\Lambda^8+\Lambda^{12}}{(1-\Lambda^4)^{15}} - 1 - 51/2\Lambda^4$. □

Proposition 8.11. $\chi_0^{\widehat{\mathbb{P}}^2, H}(5H) = \frac{1 + 21\Lambda^8 + 20\Lambda^{12} + 21\Lambda^{16} + \Lambda^{24}}{(1 - \Lambda^4)^{21}} - 1 - 33\Lambda^4$.

Proof. We use Proposition 7.13 to compute $\chi_0^{\widehat{\mathbb{P}}^2, F_+}(5H - 3E, P^r)$ for $r = 0, 2, 4$, and Proposition 7.11 to compute $\chi_0^{\widehat{\mathbb{P}}^2, F_+}(5H - 4E, P^r)$ for $r = 0, 2$. The only classes of type (0) with $\langle H, \xi \rangle \geq 0 > \langle F, \xi \rangle$ and $\delta_{\xi}^{\widehat{\mathbb{P}}^2}(5H - 3H, P^r) \neq 0$ for $r = 0, 2, 4$ or $\delta_{\xi}^{\widehat{\mathbb{P}}^2}(5H - 4H, P^r) \neq 0$ for

$r = 0, 2$ are $-2E$ and $-4E$. Adding their wallcrossing terms to the $\chi_0^{\widehat{\mathbb{P}^2}, F_+}(5H - 3E, P^r)$, $\chi_0^{\widehat{\mathbb{P}^2}, F_+}(5H - 4E, P^r)$ we get

$$\begin{aligned}\chi_0^{\widehat{\mathbb{P}^2}, H}(5H - 3E) &= \frac{1 + 6\Lambda^8 + \Lambda^{16}}{(1 - \Lambda^4)^{15}} - 1 - 27\Lambda^4 + 366\Lambda^8 - 6066\Lambda^{12} + 18917\Lambda^{16} - 33\Lambda^{20}, \\ \chi_0^{\widehat{\mathbb{P}^2}, H}(5H - 3E, P^2) &= \frac{(1 + \Lambda^4)^3}{(1 - \Lambda^4)^{13}} - 1 - 64\Lambda^4 + 2163\Lambda^8 - 32806\Lambda^{12} + 172163\Lambda^{16} \\ &\quad - 242616\Lambda^{20} + 1007\Lambda^{24}, \\ \chi_0^{\widehat{\mathbb{P}^2}, H}(5H - 3E, P^4) &= \frac{2(1 + \Lambda^4)^2}{(1 - \Lambda^4)^{11}} - 2 - 218\Lambda^4 + 10110\Lambda^8 - 170462\Lambda^{12} + 1121538\Lambda^{16} \\ &\quad - 2798450\Lambda^{20} + 2249462\Lambda^{24} - 18786\Lambda^{28}, \\ \chi_0^{\widehat{\mathbb{P}^2}, H}(5H - 4E) &= \frac{1}{(1 - \Lambda^4)^{11}} - 1 - 23\Lambda^4 + 786\Lambda^8 - 20234\Lambda^{12} + 124671\Lambda^{16} - 201885\Lambda^{20} \\ &\quad + 18372\Lambda^{24} - 21840\Lambda^{28}, \\ \chi_0^{\widehat{\mathbb{P}^2}, H}(5H - 4E, P^2) &= \frac{1}{(1 - \Lambda^4)^9} - 1 - 57\Lambda^4 + 3691\Lambda^8 - 95035\Lambda^{12} + 741175\Lambda^{16} - 2043587\Lambda^{20} \\ &\quad + 1906119\Lambda^{24} - 414993\Lambda^{28} + 295880\Lambda^{32}.\end{aligned}$$

We compute

$$R_5(x, \lambda) = -\lambda^4 x^6 + \lambda^4(1 - \lambda^4)^2(2 + \lambda^4)x^4 - 3\lambda^4(1 - \lambda^4)^4 x^2 + (1 - \lambda^4)^6.$$

Using again division with rest, we get

$$\begin{aligned}(1 - \lambda^4)^{11} &= ((\lambda^4 + 3\lambda^8)x^4 + (\lambda^4 - 8\lambda^8 + 10\lambda^{12} - 3\lambda^{20})x^2 + (1 + 4\lambda^8 - \lambda^{12})(1 - \lambda^4)^4)R_4(x, \lambda) \\ &\quad - ((\lambda^4 + 3\lambda^8)x^2 + (3 + \lambda^4)(1 - \lambda^4)^2)R_5(x, \lambda).\end{aligned}$$

Thus again we get $(1 - \Lambda^4)^{11}\chi_0^{\widehat{\mathbb{P}^2}, H}(5H)$ as the result of replacing λ by Λ and $x^r R_4(x, \lambda)$ by $\chi_0^{\widehat{\mathbb{P}^2}, H}(5H - 3E, P^r)$, $x^r R_5(x, \lambda)$ by $\chi_0^{\widehat{\mathbb{P}^2}, H}(5H - 4E, P^r)$. This gives after some computation that $\chi_0^{\widehat{\mathbb{P}^2}, H}(5H) = \frac{1+21\Lambda^8+20\Lambda^{12}+21\Lambda^{16}+\Lambda^{24}}{(1-\Lambda^4)^{21}} - 1 - 33\Lambda^4$. \square

8.4. Computer computations for larger n . We outline the computations of the PARI program to compute the $\chi_0^{\widehat{\mathbb{P}^2}, H}(nH, P^r)$, $\chi_H^{\widehat{\mathbb{P}^2}, H}(nH, P^r)$.

We have carried out these computations for $r = 0$ and $n \leq 11$, and $r \leq 16$ and $n \leq 8$.

To have effective bounds on the number of terms needed to compute we use the following remark.

Remark 8.12. For all $l \in \frac{1}{2}\mathbb{Z}$ we have

$$\frac{1}{\sinh(lh)}, \coth(lh) \in q\Lambda^{-1}\mathbb{C}[[q^{-1}\Lambda, q^4]], \quad h, \exp(lh), \widetilde{\theta}_4(h), \Lambda^2 u', h^*, M \in \mathbb{C}[[q^{-1}\Lambda, q^4]].$$

Therefore we have the following.

- (1) By Proposition 7.3, for $X = \mathbb{P}^1 \times \mathbb{P}^1$ or $\widehat{\mathbb{P}}^2$, to compute $\chi_F^{X, F+}(nF + mG, P^r)$ modulo Λ^{k+1} , it is enough to evaluate the formulas of Proposition 7.3 modulo q^{k+1} and modulo Λ^{k+1} .
- (2) By Definition 4.10 for any rational surface X and any class $\xi \in H^2(X, \mathbb{Z})$ with $\xi^2 < 0$ and any line bundle $L \in \text{Pic}(X)$, to compute $\delta_\xi^X(L, P^r)$ modulo Λ^{k+1} it is enough to evaluate the formulas of Definition 4.10 modulo q^{k+1} and modulo Λ^{k+1} .

Step 1. As mentioned above we will use Corollary 8.2 with $k = n$. The polynomials f_n, g_n, h_n, l_n , and the integers N_n, M_n of Corollary 8.2 are computed by the program as follows. Apply the Euclidean algorithm in $\mathbb{Q}(\lambda)[x^2]$ (i.e. repeated division with rest) to S_n, S_{n+1} , to find $\bar{f}_n, \bar{g}_n \in \mathbb{Q}(\lambda)[x]$ with $\bar{f}_n S_n + \bar{g}_n S_{n+1} = 1$. Choose the minimal $N_n \in \mathbb{Z}_{\geq 0}$, so that

$$f_n := \lambda(1 - \lambda^4)^{N_n} \bar{f}_n, \quad g_n := \lambda(1 - \lambda^4)^{N_n} \bar{g}_n \in \mathbb{Q}[x, \Lambda^4].$$

These exist by Proposition 6.15. Similarly h_n, l_n, M_n are computed as follows. Apply the Euclidean algorithm in $\mathbb{Q}(\lambda)[x^2]$ to R_n, R_{n+1} , to find $\bar{h}_n, \bar{l}_n \in \mathbb{Q}(\lambda)[x]$ with $\bar{h}_n R_n + \bar{l}_n R_{n+1} = 1$, and then again multiply with the minimal power $(1 - \lambda^4)^{M_n}$, to obtain.

$$h_n := (1 - \lambda^4)^{M_n} \bar{h}_n, \quad l_n := (1 - \lambda^4)^{M_n} \bar{l}_n \in \mathbb{Q}[x^2, \Lambda^4].$$

Step 2. Use Proposition 7.9 to compute $\chi_F^{X, F+}(nF, P^{2s})$ for $2s \leq \deg_x(g_n) + r$ and $\chi_0^{X, F+}(nF, P^{2s})$ for $2s \leq \deg_x(l_n) + r$. For $s = 0$, the formula is explicitly given in Proposition 7.9. For $s > 0$ we know by Proposition 7.9 that $\chi_F^{X, F+}(nF, P^{2s})$ is a polynomial in Λ^4 of degree at most s and $\chi_0^{X, F+}(nF, P^s)$ a polynomial of at most degree $s + 1$ in Λ^4 . So, using Remark 8.12, the computation is done by evaluating the formula of Proposition 7.3 as a power series in Λ, q modulo Λ^{4s+1} and q^{4s+1} or Λ^{4s+5}, q^{4s+5} respectively. As all the power series in the formula are completely explicit, this is a straightforward evaluation.

In the same way we use Proposition 7.11 to compute $\chi_F^{X, F+}(G + (n - \frac{1}{2})F, P^{2s+1})$ for $2s + 1 \leq \deg_x(f_n) + r$ and $\chi_0^{X, F+}(G + (n - \frac{1}{2})F, P^{2s})$ for $2s \leq \deg_x(h_n) + r$. By Proposition 7.11

$$\chi_F^{X, F+}(G + (n - \frac{1}{2})F, P^{2s+1}) - \frac{1}{(1 - \Lambda^4)^{2n-2s}}, \quad \chi_0^{X, F+}(G + (n - \frac{1}{2})F, P^{2s}) - \frac{1}{(1 - \Lambda^4)^{2n+1-2s}}$$

are both polynomials of degree at most $s + 1$ in Λ^4 , so, using also Remark 8.12, again they are computed by evaluating the formula of Proposition 7.3 as a power series in Λ, q modulo Λ^{4s+5} and q^{4s+5} . Again this is a straightforward evaluation.

Step 3. By the proof of Corollary 7.16 there are finitely many classes $\xi = aF - bG$ of type (0) or F on $\widehat{\mathbb{P}}^2$ with $\langle \xi, H \rangle \geq 0 > \langle \xi, F \rangle$ and $\delta_\xi^{\widehat{\mathbb{P}}^2}(nF, P^s) \neq 0$ or $\delta_\xi^{\widehat{\mathbb{P}}^2}(G + (n - \frac{1}{2})F, P^s) \neq 0$. In Remark 7.18 effective bounds for a and b are given in terms of n and s , which leave only finitely many possibilities. For all $\xi = aF - bG$, so that (a, b) satisfies these bounds, it is first checked whether indeed the criterion $-\xi^2 \leq |\langle \xi, L - K_{\widehat{\mathbb{P}}^2} \rangle| + s + 2$ for the non-vanishing of $\delta_\xi^{\widehat{\mathbb{P}}^2}(L, P^s)$ for $L = nF$ or $L = (n - 1/2)F + G$ is satisfied. If yes, $\delta_\xi^{\widehat{\mathbb{P}}^2}(L, P^s)$ is computed by evaluating the formula of Definition 4.10. By Lemma 4.15 we have that $\delta_\xi^{\widehat{\mathbb{P}}^2}(L, P^s)$ is a polynomial in Λ

of degree at most

$$a(\xi, L, X, s) := \xi^2 + 2|\langle \xi, L - K_{\widehat{\mathbb{P}^2}} \rangle| + 2s + 4,$$

so to determine $\delta_\xi^{\widehat{\mathbb{P}^2}}(L, P^s)$ we only need to compute it modulo $\Lambda^{a(\xi, L, X, s)+1}$ and thus by Remark 8.12 we only need to evaluate the formula of Definition 4.10 modulo $\Lambda^{a(\xi, L, X, s)+1}$ and $q^{a(\xi, L, X, s)+1}$, so this is again a straightforward evaluation.

Then for $c_1 = 0, F$ and $L = nF, (n - 1/2)F + G$, we compute

$$\chi_{c_1}^{\widehat{\mathbb{P}^2}, H}(L, P^s) := \chi_{c_1}^{\widehat{\mathbb{P}^2}, F_+}(L, P^s) + \frac{1}{2} \sum_{\langle \xi, H \rangle = 0 > \langle \xi, F \rangle} \delta_\xi^X(L, P^s) + \sum_{\langle \xi, H \rangle > 0 > \langle \xi, F \rangle} \delta_\xi^X(L, P^s),$$

where the sums are over all ξ of type (c_1) with $\delta_\xi^{\widehat{\mathbb{P}^2}}(L, P^s) \neq 0$.

Step 4. Finally apply Corollary 8.2 to compute

$$\begin{aligned} \chi_H^{\mathbb{P}^2, H}(nH, P^r) &= \frac{1}{\Lambda(1 - \Lambda^4)^{N_k}} \left(\chi_F^{\widehat{\mathbb{P}^2}, H}(G + (n - 1/2)F, P^r \cdot f_n(P, \Lambda)) + \chi_F^{\widehat{\mathbb{P}^2}, H}(nF, P^r \cdot g_n(P, \Lambda)) \right), \\ \chi_0^{\mathbb{P}^2, H}(nH, P^r) &= \frac{1}{(1 - \Lambda^4)^{M_k}} \left(\chi_0^{\widehat{\mathbb{P}^2}, H}(G + (n - 1/2)F, P^r \cdot h_n(P, \Lambda)) + \chi_0^{\widehat{\mathbb{P}^2}, H}(nF, P^r \cdot l_n(P, \Lambda)) \right). \end{aligned}$$

At this point all the terms on the right hand side have already been computed.

We have carried out this computation for the following cases.

- (1) For $\chi_H^{\mathbb{P}^2, H}(nH, P^r)$ with $n \equiv r \pmod{2}$, in the cases $r \leq 1, n \leq 10$ and $r \leq 16, n \leq 8$.
- (2) For $\chi_0^{\mathbb{P}^2, H}(nH, P^r)$ with r even, in the cases $r = 0, n \leq 11$ and $r \leq 15, n \leq 8$.

For the case $r = 0$ we obtain, with the notations of the introduction:

Proposition 8.13. *With the notations of Theorem 1.3 we have for $1 \leq n \leq 11$*

- (1) $\chi_0^{\mathbb{P}^2, H}(nH) = \frac{P_n(\Lambda)}{(1 - \Lambda^4)^{\binom{n+2}{2}}} - 1 - \frac{1}{2}(n^2 + 6n + 11)\Lambda^4,$
- (2) *If n is even, then $\chi_H^{\mathbb{P}^2, H}(nH) = \frac{Q_n(\Lambda)}{(1 - \Lambda^4)^{\binom{n+2}{2}}}.$*

Theorem 1.3 now follows directly from Proposition 8.13 and Proposition 2.9.

We list also the results for $\chi_H^{\mathbb{P}^2, H}(nH, P^1)$. We put

$$\begin{aligned} q_1 &= 2 - t, \quad q_3 = 2, \quad q_5 = 2 + 20t + 20t^2 + 20t^3 + 2t^4, \\ q_7 &= 2 + 80t + 770t^2 + 3080t^3 + 7580t^4 + 9744t^5 + 7580t^6 + 3080t^7 + 770t^8 + 80t^9 + 2t^{10}, \\ q_9 &= 2 + 207t + 6192t^2 + 85887t^3 + 701568t^4 + 3707406t^5 + 13050156t^6 + 31611681t^7 \\ &\quad + 53322786t^8 + 63463686t^9 + 53322786t^{10} + 31611681t^{11} + 13050156t^{12} + 3707406t^{13} \\ &\quad + 701568t^{14} + 85887t^{15} + 6192t^{16} + 207t^{17} + 2t^{18} \end{aligned}$$

Proposition 8.14. *For $1 \leq n \leq 9$ we have $\chi_{\mathbb{P}^2, dH}^H(H, P^1) = \frac{\Lambda^3 q_n(\Lambda^4)}{(1 - \Lambda^4)^{\binom{n+2}{2}-1}}.$*

We list also in the form of tables part of the results obtained for $\chi_{c_1}^{\mathbb{P}^2, H}(nH, P^r)$, with $c_1 = 0, H$ for $r > 0$. Here to simplify the expressions we only write down the results up to adding a Laurent polynomial in Λ . We define polynomials $p_{d,r}$ by the following tables.

	d=3	5	7
r=1	$2t$	$2t + 20t^2 + 20t^3 + 20t^4 + 2t^5$	$2t + 80t^2 + 770t^3 + 3080t^4 + 7580t^5 + 9744t^6 + 7580t^7 + 3080t^8 + 770t^9 + 80t^{10} + 2t^{11}$
2	$1 + t$	$1 + 6t + 25t^2 + 25t^3 + 6t^4 + t^5$	$1 + 15t + 239t^2 + 1549t^3 + 5274t^4 + 9306t^5 + 9306t^6 + 5274t^7 + 1549t^8 + 239t^9 + 15t^{10} + t^{11}$
3	$1 + t$	$8t + 24t^2 + 24t^3 + 8t^4$	$8t + 219t^2 + 1485t^3 + 5159t^4 + 9513t^5 + 9513t^6 + 5159t^7 + 1485t^8 + 219t^9 + 8t^{10}$
4	2	$2 + 14t + 32t^2 + 14t^3 + 2t^4$	$2 + 44t + 546t^2 + 2936t^3 + 7676t^4 + 10360t^5 + 7676t^6 + 2936t^7 + 546t^8 + 44t^9 + 2t^{10}$
5	2	$1 + 16t + 30t^2 + 16t^3 + t^4$	$32t + 510t^2 + 2820t^3 + 7682t^4 + 10680t^5 + 7682t^6 + 2820t^7 + 510t^8 + 32t^9$
6	$t^{-1} + 1$	$5 + 27t + 27t^2 + 5t^3$	$5 + 120t + 1209t^2 + 5075t^3 + 9975t^4 + 9975t^5 + 5075t^6 + 1209t^7 + 120t^8 + 5t^9$
7	$3 - t$	$4 + 28t + 28t^2 + 4t^3$	$1 + 99t + 1134t^2 + 4954t^3 + 10196t^4 + 10196t^5 + 4954t^6 + 1134t^7 + 99t^8 + t^9$
8		$14 + 36t + 14t^2$	$14 + 318t + 2508t^2 + 7874t^3 + 11340t^4 + 7874t^5 + 2508t^6 + 318t^7 + 14t^8$
9		$12 + 40t + 12t^2$	$6 + 276t + 2376t^2 + 7884t^3 + 11684t^4 + 7884t^5 + 2376t^6 + 276t^7 + 6t^8$
10		$t^{-1} + 31 + 31t + t^2$	$42 + 810t + 4742t^2 + 10790t^3 + 10790t^4 + 4742t^5 + 810t^6 + 42t^7$
11		$32 + 32t$	$25 + 719t + 4605t^2 + 11035t^3 + 11035t^4 + 4605t^5 + 719t^6 + 25t^7$
12		$6t^{-1} + 52 + 6t$	$132 + 1920t + 8028t^2 + 12608t^3 + 8028t^4 + 1920t^5 + 132t^6$
13		$-t^{-2} + 8t^{-1} + 50 + 8t - t^2$	$90 + 1756t + 8038t^2 + 13000t^3 + 8038t^4 + 1756t^5 + 90t^6$
14		$22t^{-1} + 57 - 21t + 7t^2 - t^3$	$t^{-1} + 407 + 4149t + 11827t^2 + 11827t^3 + 4149t^4 + 407t^5 + t^6$
15		$-4t^{-2} + 36t^{-1} + 36 - 4t$	$300 + 3964t + 12120t^2 + 12120t^3 + 3964t^4 + 300t^5$

	d=2	4	6
r=2	1	$1 + 3t + 4t^2$	$1 + 10t + 89t^2 + 272t^3 + 371t^4 + 210t^5 + 67t^6 + 4t^7$
4	t^{-1}	$2 + 5t + t^2$	$2 + 27t + 168t^2 + 370t^3 + 318t^4 + 123t^5 + 16t^6$
6		$5 + 3t$	$5 + 66t + 287t^2 + 404t^3 + 219t^4 + 42t^5 + t^6$
8		$t^{-1} + 7$	$14 + 149t + 408t^2 + 350t^3 + 98t^4 + 5t^5$
10		$4t^{-1} + 5 - t$	$42 + 288t + 468t^2 + 208t^3 + 18t^4$
12		$9t^{-1} - 1$	$t^{-1} + 116 + 462t + 388t^2 + 57t^3$
14			$8t^{-1} + 280 + 568t + 168t^2$

Theorem 8.15. *With the polynomials $p_{d,r}$ given above, we have*

- (1) *If r is even, then $\chi_0^{\mathbb{P}^2, H}(dH, P^r) \equiv \frac{p_{d,r}(\Lambda^4)}{(1 - \Lambda^4)^{\binom{d+2}{2} - r}}$.*
- (2) *If d and r are both odd, then $\chi_H^{\mathbb{P}^2, H}(dH, P^r) \equiv \frac{\Lambda^{-1} p_{d,r}(\Lambda^4)}{(1 - \Lambda^4)^{\binom{d+2}{2} - r}} = \frac{\Lambda^{d^2 - 2r} p_{d,r}(\Lambda^{-4})}{(1 - \Lambda^4)^{\binom{d+2}{2} - r}}$.*
- (3) *If d and r are both even, then $\chi_H^{\mathbb{P}^2, H}(dH, P^r) \equiv \frac{\Lambda^{d^2 - 2r - 1} p_{d,r}(\Lambda^{-4})}{(1 - \Lambda^4)^{\binom{d+2}{2} - r}}$.*

8.5. Invariants of blowups of the plane. We want to apply the above results to compute K -theoretic Donaldson invariants of blowups of \mathbb{P}^2 in a finite number of points.

Remark 8.16. Let X_r be the blowup of \mathbb{P}^2 in r general points and let $E := E_1 + \dots + E_r$ be the sum of the exceptional divisors. By definition we have for $c_1 = 0, H$ that $\chi_{c_1+E}^{X_r, H}(nH - E) = \Lambda^r \chi_{c_1}^{\mathbb{P}^2, H}(nH, P^r)$. By Lemma 8.4 we have therefore $\chi_{c_1+E}^{X_r, \omega}(nH - E) \equiv \Lambda^r \chi_{c_1}^{\mathbb{P}^2, H}(nH, P^r)$ for all classes $\omega = H - \sum_{i=1}^r a_i E_i$ on X_r with $\langle \omega, K_{X_r} \rangle < 0$ and $0 \leq a_i < \frac{1}{\sqrt{r}}$ for all i . Therefore the formulas of Theorem 8.15 also give the $\chi_{c_1+E}^{X_r, \omega}(nH - E)$.

By Theorem 6.7, and using Lemma 8.4, we can, from the $\chi_0^{\mathbb{P}^2, H}(nH, P^r)$, $\chi_H^{\mathbb{P}^2, H}(nH, P^r)$, readily compute the generating functions of K -theoretical Donaldson invariants $\chi_{c_1}^{X, \omega}(L)$ for any blowup X of \mathbb{P}^2 in finitely many points, for any $c_1, L \in \text{Pic}(X)$, and for any ω close to H , up to addition of a Laurent polynomial. In particular we can readily apply this computation to the tables of the $\chi_0^{\mathbb{P}^2, H}(nH, P^r)$, $\chi_H^{\mathbb{P}^2, H}(nH, P^r)$ of Theorem 8.15 above. We will only write down the result in one simple case. We take X_s the blowup of \mathbb{P}^2 in s points, and let again $E = \sum_{i=1}^s E_i$ be the sum of the exceptional divisors, let $L := dH - 2E$ and consider the cases $c_1 = 0$, $c_1 = E$, $c_1 = H$ and $c_1 = K_{X_s}$.

We define polynomials $q_{d,s}$ by the following table,

	d=3	4	5	6
s=1	1	$1 + 3t^2$	$1 + 15t^2 + 10t^3 + 6t^4$	$1 + 46t^2 + 104t^3 + 210t^4 + 105t^5 + 43t^6 + 3t^7$
s=2		$1 + t^2$	$1 + 10t^2 + 4t^3 + t^4$	$1 + 37t^2 + 70t^3 + 105t^4 + 34t^5 + 9t^6$
s=3		1	$1 + 6t^2 + t^3$	$1 + 29t^2 + 44t^3 + 45t^4 + 8t^5 + t^6$
s=4			$1 + 3t^2$	$1 + 22t^2 + 25t^3 + 15t^4 + t^5$
s=5			$1 + t^2$	$1 + 16t^2 + 12t^3 + 3t^4$
s=6			1	$1 + 11t^2 + 4t^3$
s=7				$1 + 7t^2$

and polynomials $r_{d,s}$ by the following table.

	d=4	6
s=1	$1 + 3t$	$1 + 24t + 105t^2 + 161t^3 + 168t^4 + 43t^5 + 10t^6$
s=2	$1 + t$	$1 + 21t + 71t^2 + 90t^3 + 63t^4 + 9t^5 + t^6$
s=3	1	$1 + 18t + 45t^2 + 45t^3 + 18t^4 + t^5$
s=4		$1 + 15t + 26t^2 + 19t^3 + 3t^4$
s=5		$1 + 12t + 13t^2 + 6t^3$
s=6		$1 + 9t + 5t^2 + t^3$
s=7		$1 + 6t + t^2$
s=8		$1 + 3t$

Proposition 8.17. *Let X_s be the blowup of \mathbb{P}^2 in r general points with exceptional divisors E_1, \dots, E_s , and write $E = \sum_{i=1}^s E_i$. With the $q_{d,s}$ and $r_{d,s}$ given by the above tables we get*

$$\chi_0^{X, H}(dH - 2E) \equiv \frac{q_{d,s}(\Lambda^4)}{(1 - \Lambda^4)^{\binom{d+2}{2} - 3s}}$$

$$\begin{aligned}
\chi_{K_{X_s}}^{X,H}(dH - 2E) &\equiv \frac{\Lambda^{d^2-1-3s} q_{d,s}(\frac{1}{\Lambda^4})}{(1-\Lambda^4)^{\binom{d+2}{2}-3s}}, & d \text{ even} \\
\chi_H^{X,H}(dH - 2E) &\equiv \frac{\Lambda^3 r_{d,s}(\Lambda^4)}{(1-\Lambda^4)^{\binom{d+2}{2}-3s}}, & d \text{ even} \\
\chi_E^{X,H}(dH - 2E) &\equiv \begin{cases} \frac{\Lambda^{d^2-1-3s} q_{d,s}(\frac{1}{\Lambda^4})}{(1-\Lambda^4)^{\binom{d+2}{2}-3s}} & d \text{ odd} \\ \frac{\Lambda^{d^2-4-3s} r_{d,s}(\frac{1}{\Lambda^4})}{(1-\Lambda^4)^{\binom{d+2}{2}-3s}} & d \text{ even} \end{cases}
\end{aligned}$$

The same formulas also apply with $\chi_{c_1}^{X,H}(dH - 2E)$ replaced by $\chi_{c_1}^{X,\omega}(dH - 2E)$, with $\omega = H - a_1 E_1 - \dots - a_r E_s$ and $0 \leq a_i \leq \sqrt{s}$ for all i .

Proof. Recall that $R_3 = -\lambda^4 x^2 + (1 - \lambda^4)^2$, $S_3 = \lambda(x^2 - (1 - \lambda^4)^2)$. Noting that $K_{X_s} \equiv H + E \pmod{2H^2(X, \mathbb{Z})}$, we get by Theorem 6.7 that

$$\begin{aligned}
\chi_0^{X_s,H}(dH - 2E) &= \chi_0^{\mathbb{P}^2,H}(dH, (-\Lambda^4 P^2 + 1 - \Lambda^4)^2)^s, \\
\chi_H^{X_s,H}(dH - 2E) &= \chi_H^{\mathbb{P}^2,H}(dH, (-\Lambda^4 P^2 + 1 - \Lambda^4)^2)^s, \\
\chi_E^{X_s,H}(dH - 2E) &= \Lambda^s \chi_0^{\mathbb{P}^2,H}(dH, (P^2 - (1 - \Lambda^4)^2)^s), \\
\chi_{K_{X_s}}^{X_s,H}(dH - 2E) &= \Lambda^s \chi_H^{\mathbb{P}^2,H}(dH, (P^2 - (1 - \Lambda^4)^2)^s).
\end{aligned}$$

Now we just put the values of the tables of Theorem 8.15 into these formulas. \square

8.6. Symmetries from Cremona transforms.

Remark 8.18. The Conjecture 1.7 will often predict a symmetry for the polynomials $P_{c_1,L}^X(\Lambda)$. Assume Conjecture 1.7. Then we have the following.

- (1) If $c_1 \equiv L + K_X - c_1 \pmod{2H^2(X, \mathbb{Z})}$, then $P_{c_1,L}^X(\Lambda) = \Lambda^{L^2+8-K_X^2} P_{c_1,L}^X(\frac{1}{\Lambda})$.
- (2) More generally let X be the blowup of \mathbb{P}^2 in n points, with exceptional divisors E_1, \dots, E_n , $L = dH - a_1 E_1 - \dots - a_n E_n$. If σ is a permutation of $\{1, \dots, n\}$, we write $\sigma(L) := dH - a_{\sigma(1)} E_1 - \dots - a_{\sigma(n)} E_n$. Then $\chi_{c_1}^{X,H}(L) = \chi_{\sigma(c_1)}^{X,H}(\sigma(L))$.

Thus, if there is a σ with $L = \sigma(L)$ and $\sigma(c_1) \equiv L + K_X - c_1 \pmod{2H^2(X, \mathbb{Z})}$, then $P_{c_1,L}^X(\Lambda) = \Lambda^{L^2+8-K_X^2} P_{c_1,L}^X(\frac{1}{\Lambda})$.

Other symmetries come from the Cremona transform of the plane, which we briefly review. Let p_1, p_2, p_3 be three general points in \mathbb{P}^2 . For $i = 1, 2, 3$ let L_i the line through p_i, p_j where $\{i, j, k\} = \{1, 2, 3\}$. Let X be the blowup of \mathbb{P}^2 in p_1, p_2, p_3 , with exceptional divisors E_1, E_2, E_3 , and let $\overline{E}_1, \overline{E}_2, \overline{E}_3$ be the strict transforms of the lines L_1, L_2, L_3 . The \overline{E}_i are disjoint (-1) curves which can be blown down to obtain another projective plane $\overline{\mathbb{P}}^2$. Let H (resp. \overline{H}) be the pullback of the hyperplane class from \mathbb{P}^2 (resp. $\overline{\mathbb{P}}^2$) to X . Then $H^2(X, \mathbb{Z})$ has two different bases H, E_1, E_2, E_3 and $\overline{H}, \overline{E}_1, \overline{E}_2, \overline{E}_3$, which are related by the formula

$$dH - a_1 E_1 - a_2 E_2 - a_3 E_3 = (2d - a_1 - a_2 - a_3) \overline{H} - (d - a_2 - a_3) \overline{E}_1 - (d - a_1 - a_3) \overline{E}_2 - (d - a_1 - a_2) \overline{E}_3.$$

Note that this description is symmetric under exchanging the role of H, E_1, E_2, E_3 and $\overline{H}, \overline{E}_1, \overline{E}_2, \overline{E}_3$. Let $c_1 \in H^2(X, \mathbb{Z})$. If $\langle c_1, K_X \rangle$ is even, then it is easy to see that $\bar{c}_1 \equiv c_1 \pmod{2H^2(X, \mathbb{Z})}$, but if $\langle c_1, K_X \rangle$ is odd, then $\bar{c}_1 \equiv K_X - c_1 \pmod{2H^2(X, \mathbb{Z})}$. For a class $L = dH - a_1E_1 - a_2E_2 - a_3E_3 \in H^2(X, \mathbb{Z})$ we denote $\bar{L} = d\overline{H} - a_1\overline{E}_1 - a_2\overline{E}_2 - a_3\overline{E}_3$. Then it is clear from the definition that $\chi_{c_1}^{X,H}(L) = \chi_{\bar{c}_1}^{X,\overline{H}}(\bar{L})$, and by Lemma 8.4 we get $\chi_{\bar{c}_1}^{X,\overline{H}}(\bar{L}) \equiv \chi_{\bar{c}_1}^{X,H}(\bar{L})$. If σ is a permutation of $\{1, 2, 3\}$ and we denote $\sigma(L) := dH - a_{\sigma_1}E_1 - a_{\sigma_2}E_2 - a_{\sigma_3}E_3$, If $\sigma(L) = L$, then it is clear that $\chi_{c_1}^{X,H}(L) = \chi_{\sigma(c_1)}^{X,H}(L)$.

Now assume $d = a_1 + a_2 + a_3$. Then $\bar{L} = L$, so that $\chi_{c_1}^{X,H}(L) \equiv \chi_{\bar{c}_1}^{X,H}(L)$.

Assume now $\langle c_1, K_X \rangle$ is odd. Assuming also Conjecture 1.7, the polynomials $P_{c_1,L}^X \in \Lambda^{-c_1^2\mathbb{Z}[\Lambda^4]}$ mentioned there satisfy

- (1) $P_{c_1,L}^X(\Lambda) = P_{K_X - c_1,L}^X(\Lambda) = \Lambda^{L^2+8-K_X^2} P_{L+K_X-c_1}^X(\frac{1}{\Lambda}) = \Lambda^{L^2+8-K_X^2} P_{L-c_1}^X(\frac{1}{\Lambda})$.
- (2) Thus if there is a permutation σ of $\{1, 2, 3\}$ with $\sigma(L) = L$ and $c_1 \equiv L - \sigma(c_1) \pmod{2H^2(X, \mathbb{Z})}$, or with $\sigma(L) = L$ and $c_1 \equiv L + K_X - \sigma(c_1) \pmod{2H^2(X, \mathbb{Z})}$. Then we have the symmetries

$$P_{c_1,L}^X(\Lambda) = \Lambda^{L^2+8-K_X^2} P_{c_1,L}^X(\frac{1}{\Lambda}) = P_{K_X - c_1,L}^X(\Lambda) = \Lambda^{L^2+8-K_X^2} P_{K_X - c_1,L}^X(\frac{1}{\Lambda}).$$

We check these predictions in a number of cases. Let $L_5 = 5H - 2E_1 - 2E_2 - E_3$, $L_6 = 6H - 2E_1 - 2E_2 - 2E_3$, $L_7 = 7H - 3E_1 - 2E_2 - 2E_3$. We find

$$\begin{aligned} \chi_{K_X+E_2}^{X,H}(L_5) &\equiv \chi_{E_2}^{X,H}(L_5) \equiv \frac{5\Lambda^5 + 6\Lambda^9 + 5\Lambda^{13}}{(1 - \Lambda^4)^{14}}, \\ \chi_H^{X,H}(L_6) &\equiv \chi_{E_1+E_2+E_3}^{X,H}(L_6) \equiv \frac{\Lambda^3 + 18\Lambda^7 + 45\Lambda^{11} + 45\Lambda^{15} + 18\Lambda^{19} + \Lambda^{23}}{(1 - \Lambda^4)^{19}}, \\ \chi_{K_X-E_3}^{X,H}(L_6) &\equiv \chi_{E_3}^{X,H}(L_6) \equiv \frac{8\Lambda^5 + 26\Lambda^9 + 60\Lambda^{13} + 26\Lambda^{17} + 8\Lambda^{21}}{(1 - \Lambda^4)^{19}}, \\ \chi_{K_X-E_3}^{X,H}(L_7) &\equiv \chi_{E_3}^{X,H}(L_7) \equiv \frac{11\Lambda^5 + 61\Lambda^9 + 265\Lambda^{13} + 350\Lambda^{17} + 265\Lambda^{21} + 61\Lambda^{25} + 11\Lambda^{29}}{(1 - \Lambda^4)^{24}}. \end{aligned}$$

9. THE INVARIANTS OF $\mathbb{P}^1 \times \mathbb{P}^1$

In this section we will use the results of the previous section to compute the K -theoretic Donaldson invariants of $\mathbb{P}^1 \times \mathbb{P}^1$.

9.1. A structural result. First we will show analogously to Theorem 8.7 that all the generating functions $\chi_{c_1}^{\mathbb{P}^1 \times \mathbb{P}^1, \omega}(L, P^r)$ are rational functions.

Lemma 9.1. *Let $c_1 \in H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z})$. Let L be a line bundle on $\mathbb{P}^1 \times \mathbb{P}^1$ with $\langle L, c_1 \rangle + r$ even. Let ω be an ample classes on $\mathbb{P}^1 \times \mathbb{P}^1$. Then $\chi_{c_1}^{\mathbb{P}^1 \times \mathbb{P}^1, \omega}(L, P^r) \equiv \chi_{c_1}^{\mathbb{P}^1 \times \mathbb{P}^1, F+G}(L, P^r)$.*

Proof. We write $L = nF + mG$. By exchanging the role of F and G if necessary we can write $\omega = G + \alpha F$, with $1 \leq \alpha$. We have to show that there are only finitely many classes ξ of type

(c_1) with $\langle \xi, (F+G) \rangle \leq 0 \leq \langle \xi, \omega \rangle$ and $\delta_\xi^X(L, P^r) \neq 0$. Such a class is of the form $\xi = aG - bF$, with $a, b \in \mathbb{Z}_{>0}$ satisfying

$$a \leq b, \quad \alpha a \geq b, \quad 2ab \leq |a(n+2) - b(m+2)| + r + 2.$$

This gives

$$2ab \leq a|n+2| + b|m+2| + r + 2 \leq b|n+m+4| + r + 2.$$

Thus we get $a \leq \frac{|n+m+4|}{2} + \frac{r}{2} + 1$. Therefore a is bounded and by $\alpha a \geq b$ also b is bounded. Therefore there are only finitely many possible classes ξ . \square

We use the fact that the blowup $\tilde{\mathbb{P}}^2$ of \mathbb{P}^2 in two different points is also the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ in a point. We can identify the classes as follows. Let H be the hyperplane class on \mathbb{P}^2 and let E_1, E_2 be the exceptional divisors of the double blowup of \mathbb{P}^2 . Let F, G be the fibres of the two different projections of $\mathbb{P}^1 \times \mathbb{P}^1$ to its factors, and let E be the exceptional divisor of the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$. Then on $\tilde{\mathbb{P}}^2$ we have the identifications

$$\begin{aligned} F &= H - E_1, & G &= H - E_2, & E &= H - E_1 - E_2, \\ H &= F + G - E, & E_1 &= G - E, & E_2 &= F - E. \end{aligned}$$

Theorem 9.2. *Let $c_1 \in \{0, F, G, F+G\}$. Let L be a line bundle on $\mathbb{P}^1 \times \mathbb{P}^1$ with $\langle L, c_1 \rangle$ even. Let $r \in \mathbb{Z}_{\geq 0}$ with $\langle L, c_1 \rangle + r$ even. There exists a polynomial $p_{c_1, L, r}^{\mathbb{P}^1 \times \mathbb{P}^1}(t)$ and an integer $N_{c_1, L, r}^{\mathbb{P}^1 \times \mathbb{P}^1}$, such that for all ample classes ω on $\mathbb{P}^1 \times \mathbb{P}^1$, we have*

$$\chi_{c_1}^{\mathbb{P}^1 \times \mathbb{P}^1, \omega}(L, P^r) \equiv \frac{p_{c_1, L, r}^{\mathbb{P}^1 \times \mathbb{P}^1}(\Lambda^4)}{\Lambda^{c_1^2}(1 - \Lambda^4)^{N_{c_1, L, r}^{\mathbb{P}^1 \times \mathbb{P}^1}}}.$$

Proof. Note that on $\tilde{\mathbb{P}}^2$ we have $F+G = 2H - E_1 - E_2$. We write $L = nF + mG$, with $n, m \in \mathbb{Z}$. Then on $\tilde{\mathbb{P}}^2$ we have $L = (n+m)H - nE_1 - mE_2$. By Theorem 6.7 we have therefore

$$\begin{aligned} \chi_0^{\tilde{\mathbb{P}}^2, H}(nF + mG, P^r) &= \chi_0^{\mathbb{P}^2, H}((n+m)H, P^r \cdot R_{n+1}(P, \Lambda)R_{m+1}(P, \Lambda)), \\ \chi_F^{\tilde{\mathbb{P}}^2, H}(nF + mG, P^r) &= \chi_H^{\mathbb{P}^2, H}((n+m)H, P^r \cdot S_{n+1}(P, \Lambda)R_{m+1}(P, \Lambda)), \\ \chi_G^{\tilde{\mathbb{P}}^2, H}(nF + mG, P^r) &= \chi_H^{\mathbb{P}^2, H}((n+m)H, P^r \cdot R_{n+1}(P, \Lambda)S_{m+1}(P, \Lambda)), \\ \chi_{F+G}^{\tilde{\mathbb{P}}^2, H}(nF + mG, P^r) &= \chi_0^{\mathbb{P}^2, H}((n+m)H, P^r \cdot S_{n+1}(P, \Lambda)S_{m+1}(P, \Lambda)). \end{aligned}$$

As $R_n(P, \Lambda) \in \mathbb{Z}[P^2, \Lambda^4]$, $S_n(P, \Lambda) \in \Lambda\mathbb{Z}[P, \Lambda^4]$, we see by Theorem 8.3 that for $c_1 = 0, F, G, F+G$ we can write

$$\chi_{c_1}^{\tilde{\mathbb{P}}^2, H}(nF + mG, P^r) \equiv \frac{p_{c_1, nF+mG, r}^{\mathbb{P}^1 \times \mathbb{P}^1}(\Lambda^4)}{\Lambda^{c_1^2}(1 - \Lambda^4)^{N_{c_1, nF+mG, r}^{\mathbb{P}^1 \times \mathbb{P}^1}}},$$

with $p_{c_1, nF+mG, r}^{\mathbb{P}^1 \times \mathbb{P}^1} \in \mathbb{Q}[t]$ and $N_{c_1, nF+mG, r}^{\mathbb{P}^1 \times \mathbb{P}^1} \in \mathbb{Z}_{\geq 0}$. As on $\tilde{\mathbb{P}}^2$ we have $F+G = 2H - E_1 - E_2$, we get by Theorem 8.7 that again for $c_1 = 0, F, G, F+G$ we have $\chi_{c_1}^{\tilde{\mathbb{P}}^2, F+G}(nF + mG, P^r) \equiv \chi_{c_1}^{\tilde{\mathbb{P}}^2, H}(nF + mG, P^r)$. Finally by the blowdown formula Theorem 6.7 we have $\chi_{c_1}^{\tilde{\mathbb{P}}^2, F+G}(nF + mG, P^r) = \chi_{c_1}^{\mathbb{P}^1 \times \mathbb{P}^1, F+G}(nF + mG, P^r)$. \square

9.2. Computations for $L = d(F + G)$. We will compute $\chi_{c_1}^{\mathbb{P}^1 \times \mathbb{P}^1, F+G}(d(F + G))$ for $d \leq 7$ and $c_1 = 0, F, G, F + G$. Obviously by symmetry $\chi_G^{\mathbb{P}^1 \times \mathbb{P}^1, F+G}(d(F + G)) = \chi_F^{\mathbb{P}^1 \times \mathbb{P}^1, F+G}(d(F + G))$, and furthermore we have just seen that $\chi_{c_1}^{\mathbb{P}^1 \times \mathbb{P}^1, \omega}(d(F + G)) \equiv \chi_{c_1}^{\mathbb{P}^1 \times \mathbb{P}^1, F+G}(d(F + G))$ for any ample class ω on $\mathbb{P}^1 \times \mathbb{P}^1$. We use a different strategy than in the proof of Theorem 9.2, which is computationally more tractable, and allows us to compute $\chi_{c_1}^{\mathbb{P}^1 \times \mathbb{P}^1, F+G}(d(F + G))$ for $d \leq 7$, using only the $\chi_H^{\mathbb{P}^2, H}(nH, P^r)$, for $1 \leq n \leq 8, 0 \leq r \leq 16$ already computed.

Step 1. By $F = H - E_1, G = H - E_2, E = H - E_1 - E_2$, and thus $d(F + G) - dE = dH$, and Theorem 6.7 we have

$$\begin{aligned} \chi_E^{\tilde{\mathbb{P}}^2, H}(d(F + G) - dE, P^r) &= \chi_{H-E_1-E_2}^{\tilde{\mathbb{P}}^2, H}(dH, P^r) = \Lambda^2 \chi_H^{\mathbb{P}^2, H}(dH, P^r), \\ \chi_E^{\tilde{\mathbb{P}}^2, H}(d(F + G) - (d-1)E, P^r) &= \chi_{H-E_1-E_2}^{\tilde{\mathbb{P}}^2, H}((d+1)H - E_1 - E_2, P^r) \\ &= \Lambda^2 \chi_H^{\mathbb{P}^2, H}((d+1)H, P^{r+2}), \\ \chi_F^{\tilde{\mathbb{P}}^2, H}(d(F + G) - dE, P^r) &= \chi_{H-E_1}^{\mathbb{P}^2, H}(dH, P^r) = \Lambda \chi_H^{\mathbb{P}^2, H}(dH, P^r), \\ \chi_F^{\tilde{\mathbb{P}}^2, H}(d(F + G) - (d-1)E, P^r) &= \chi_{H-E_1}^{\tilde{\mathbb{P}}^2, H}((d+1)H - E_1 - E_2, P^r) \\ &= \Lambda(1 - \Lambda^4) \chi_H^{\mathbb{P}^2, H}((d+1)H, P^{r+1}), \\ \chi_{F+G-E}^{\tilde{\mathbb{P}}^2, H}(d(F + G) - dE, P^r) &= \chi_H^{\mathbb{P}^2, H}(dH, P^r), \\ \chi_{F+G-E}^{\tilde{\mathbb{P}}^2, H}(d(F + G) - (d-1)E, P^r) &= \chi_H^{\tilde{\mathbb{P}}^2, H}((d+1)H - E_1 - E_2, P^r) \\ &= (1 - \Lambda^4)^2 \chi_H^{\mathbb{P}^2, H}((d+1)H, P^r), \end{aligned}$$

where we have used that $S_1(P, \Lambda) = \Lambda, S_2(P, \Lambda) = P\Lambda$ and $R_2(P, \Lambda) = (1 - \Lambda^4)$.

The $\chi_H^{\mathbb{P}^2, H}(nH, P^s)$ have been computed for $n \leq 8$ and $s \leq 16$. In the tables above they are only listed for $n \leq 7$ and only up to adding a Laurent polynomial in Λ , so as to give them a particularly simple form, but they have been computed precisely. Thus in the range $d \leq 7$ and $r \leq 14$, the all the invariants on the left hand side of the formulas of Step 1 have been computed.

Step 2. For $d \leq 7$ we compute

$$\begin{aligned} (1) \quad \chi_E^{\tilde{\mathbb{P}}^2, F+G}(d(F + G) - dE, P^r) &= \Lambda^2 \chi_H^{\mathbb{P}^2, H}(dH, P^r) + \sum_{\xi} \delta_{\xi}^{\tilde{\mathbb{P}}^2}(dH, P^r), \\ \chi_E^{\tilde{\mathbb{P}}^2, F+G}(d(F + G) - (d-1)E, P^r) &= \Lambda^2 \chi_H^{\mathbb{P}^2, H}((d+1)H, P^{r+2}) \\ &\quad + \sum_{\xi} \delta_{\xi}^{\tilde{\mathbb{P}}^2}((d+1)H - E_1 - E_2, P^r), \\ (2) \quad \chi_F^{\tilde{\mathbb{P}}^2, F+G}(d(F + G) - dE, P^r) &= \Lambda \chi_H^{\mathbb{P}^2, H}(dH, P^r) + \sum_{\xi} \delta_{\xi}^{\tilde{\mathbb{P}}^2}(dH, P^r), \\ \chi_F^{\tilde{\mathbb{P}}^2, F+G}(d(F + G) - (d-1)E, P^r) &= \Lambda(1 - \Lambda^4) \chi_H^{\mathbb{P}^2, H}((d+1)H, P^{r+1}) \\ &\quad + \sum_{\xi} \delta_{\xi}^{\tilde{\mathbb{P}}^2}((d+1)H - E_1 - E_2, P^r), \end{aligned}$$

$$\begin{aligned}
(3) \quad \chi_{F+G-E}^{\tilde{\mathbb{P}}^2, F+G}(d(F+G) - dE, P^r) &= \chi_H^{\mathbb{P}^2, H}(dH, P^r) + \sum_{\xi} \delta_{\xi}^{\tilde{\mathbb{P}}^2}(dH, P^r), \\
\chi_{F+G-E}^{\tilde{\mathbb{P}}^2, F+G}(d(F+G) - (d-1)E, P^r) &= (1 - \Lambda^4)^2 \chi_H^{\mathbb{P}^2, H}((d+1)H, P^r) \\
&\quad + \sum_{\xi} \delta_{\xi}^{\tilde{\mathbb{P}}^2}((d+1)H - E_1 - E_2, P^r).
\end{aligned}$$

Here the sums are over all classes $\xi \in H^2(\tilde{\mathbb{P}}^2, \mathbb{Z})$ with $\langle H, \xi \rangle \leq 0 \leq \langle (2H - E_1 - E_2), \xi \rangle$ (but at least one of the inequalities is strict) and $\delta_{\xi}^{\tilde{\mathbb{P}}^2} \neq \emptyset$, and the summand is $\delta_{\xi}^{\tilde{\mathbb{P}}^2}(dH, P^r)$, $\delta_{\xi}^{\tilde{\mathbb{P}}^2}((d+1)H - E_1 - E_2, P^r)$, if both inequalities are strict and $\frac{1}{2}\delta_{\xi}^{\tilde{\mathbb{P}}^2}(dH, P^r)$, $\frac{1}{2}\delta_{\xi}^{\tilde{\mathbb{P}}^2}((d+1)H - E_1 - E_2, P^r)$ if one of them is an equality. (Note that we can exclude the ξ with $\langle \xi, H \rangle = 0 = \langle \xi, 2H - E_1 - E_2 \rangle$, because with ξ also $-\xi$ will fulfil this property and $\delta_{\xi}^{\tilde{\mathbb{P}}^2}(L, P^r) = -\delta_{-\xi}^{\tilde{\mathbb{P}}^2}(L, P^r)$.)

In (1) these are classes of type $(E = H - E_1 - E_2)$, in (2) of type $(F = H - E_1)$ and in (3) of type $(F + G - E = H)$. By Lemma 8.4 there are finitely many such classes. In fact in the notations of the proof of Lemma 8.4 we have

$$n = 2, \quad \epsilon = \frac{1}{2}, \quad \delta = \frac{1}{4}, \quad |m_1 + 1| = |m_2 + 1| = 1.$$

Thus if $\xi = aH - b_1E_1 - b_2E_2$ is such a class, then we get by (8.6) that $|b_1| + |b_2| \leq 4(|d+3| + r + 4)$ and $0 < a \leq \frac{1}{2}(|b_1| + |b_2|)$. For all ξ satisfying these bounds it is first checked whether indeed the criterion of Lemma 4.15(2) for the non-vanishing of the wallcrossing term $\delta_{\xi}^{\tilde{\mathbb{P}}^2}(dH, P^r)$, $\delta_{\xi}^{\tilde{\mathbb{P}}^2}((d+1)H - E_1 - E_2, P^r)$ is fulfilled, if yes we compute the wallcrossing term, we again use that by Lemma 4.15 $\delta_{\xi, d}^X(L, P^r) = 0$ unless $d \leq a_{\xi, L, X} := \xi^2 + 2|\langle \xi, L - K_X \rangle| + 2r + 4$. Thus, also using Remark 8.12 it is enough evaluate the formula of Definition 4.10 modulo $q^{a_{\xi, L, X}}$ and $\Lambda^{a_{\xi, L, X}}$. This is again a finite evaluation.

Step 3. By Theorem 6.7 we have

$$\begin{aligned}
\chi_E^{\tilde{\mathbb{P}}^2, F+G}(d(F+G) - dE, P^r) &= \chi_0^{\mathbb{P}^1 \times \mathbb{P}^1, F+G}(d(F+G), S_{d+1}(P, \Lambda)P^r), \\
\chi_E^{\tilde{\mathbb{P}}^2, F+G}(d(F+G) - (d-1)E, P^r) &= \chi_0^{\mathbb{P}^1 \times \mathbb{P}^1, F+G}(d(F+G), S_d(P, \Lambda)P^r), \\
\chi_F^{\tilde{\mathbb{P}}^2, F+G}(d(F+G) - dE, P^r) &= \chi_F^{\mathbb{P}^1 \times \mathbb{P}^1, F+G}(d(F+G), R_{d+1}(P, \Lambda)P^r), \\
\chi_F^{\tilde{\mathbb{P}}^2, F+G}(d(F+G) - (d-1)E, P^r) &= \chi_F^{\mathbb{P}^1 \times \mathbb{P}^1, F+G}(d(F+G), R_d(P, \Lambda)P^r), \\
\chi_{F+G-E}^{\tilde{\mathbb{P}}^2, F+G}(d(F+G) - dE, P^r) &= \chi_{F+G}^{\mathbb{P}^1 \times \mathbb{P}^1, F+G}(d(F+G), S_{d+1}(P, \Lambda)P^r), \\
\chi_{F+G-E}^{\tilde{\mathbb{P}}^2, F+G}(d(F+G) - (d-1)E, P^r) &= \chi_{F+G}^{\mathbb{P}^1 \times \mathbb{P}^1, F+G}(d(F+G), S_d(P, \Lambda)P^r).
\end{aligned}$$

By Remark 8.1 there exist polynomials $f_d \in \mathbb{Q}[x, \Lambda^4]$, $g_d \in \mathbb{Q}[x, \Lambda^4]$ with $f_d S_d(x, \lambda) + g_d S_{d+1}(x, \lambda) = \lambda(1 - \lambda^4)^{N_d}$, and $h_d \in \mathbb{Q}[x, \Lambda^4]$, $l_d \in \mathbb{Q}[x, \Lambda^4]$ with $h_d R_d(x, \lambda) + l_d R_{d+1}(x, \lambda) = (1 - \lambda^4)^{M_d}$. For $d \leq 7$ we see that f_d, h_d are polynomials in x of degree at most 14, and g_d, l_d are polynomials in x of degree at most 11.

Thus we get

$$\begin{aligned}\chi_0^{\mathbb{P}^1 \times \mathbb{P}^1, F+G}(d(F+G)) &= \frac{1}{\Lambda(1-\Lambda^4)^{M_d}} \left(\chi_E^{\tilde{\mathbb{P}}^2, F+G}(d(F+G) - (d-1)E, f_d(P, \Lambda)) \right. \\ &\quad \left. + \chi_E^{\tilde{\mathbb{P}}^2, F+G}(d(F+G) - dE, g_d(P, \Lambda)) \right), \\ \chi_F^{\mathbb{P}^1 \times \mathbb{P}^1, F+G}(d(F+G)) &= \frac{1}{(1-\Lambda^4)^{N_d}} \left(\chi_F^{\tilde{\mathbb{P}}^2, F+G}(d(F+G) - (d-1)E, h_d(P, \Lambda)) \right. \\ &\quad \left. + \chi_F^{\tilde{\mathbb{P}}^2, F+G}(d(F+G) - dE, l_d(P, \Lambda)) \right), \\ \chi_{F+G}^{\mathbb{P}^1 \times \mathbb{P}^1, F+G}(d(F+G)) &= \frac{1}{\Lambda(1-\Lambda^4)^{M_d}} \left(\chi_{F+G-E}^{\tilde{\mathbb{P}}^2, F+G}(d(F+G) - (d-1)E, f_d(P, \Lambda)) \right. \\ &\quad \left. + \chi_{F+G-E}^{\tilde{\mathbb{P}}^2, F+G}(d(F+G) - dE, g_d(P, \Lambda)) \right).\end{aligned}$$

All these computations are carried out with a Pari program. Finally we arrive at the following result.

Theorem 9.3. *With the notation of Theorem 1.5.*

- (1) $\chi_0^{\mathbb{P}^1 \times \mathbb{P}^1, F+G}(dF + dG) = \frac{p_d(\Lambda^4)}{(1-\Lambda^4)^{(d+1)^2}} - 1 - (d^2 + 4d + 5)\Lambda^4$ for $1 \leq d \leq 7$.
- (2) $\chi_{F+G}^{\mathbb{P}^1 \times \mathbb{P}^1, F+G}(dF + dG) = \frac{q_d(\Lambda^4)}{(1-\Lambda^4)^{(d+1)^2}} - \Lambda^2$ for $1 \leq d \leq 7$.
- (3) $\chi_F^{\mathbb{P}^1 \times \mathbb{P}^1, F+G}(dF + dG) = \frac{r_d(\Lambda^4)}{(1-\Lambda^4)^{(d+1)^2}}$ for $d = 2, 4, 6$.

Theorem 1.5 follows from this and Proposition 2.9.

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